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ON FLUCTUATIONS OF EIGENVALUES OF RANDOM HERMITIAN MATRICES

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1. Introduction. Let $\mathscr{H}_N$ be the set of all $N \times N$ hermitian matrices, $\Phi = (\Phi_{ij})_{1 \leqslant i,j \leqslant N}$, and let $\mathbb{P}_{2\ell}^+$ denote the set of all polynomials of degree 2ℓ with positive leading coefficient. On $\mathscr{H}_N$ we consider the probability measure

$$d\mu_{N,M}(\Phi) = \frac{1}{\mathscr{Z}_{N,M}^2} \exp(-M \operatorname{Tr} V(\Phi)) d\Phi, \tag{1.1}$$

where $V \in \mathbb{P}_{2\ell}^+$, $\ell \geqslant 1$, $M > 0$, and

$$d\Phi = \prod_{1 \leqslant i < j \leqslant N} d \operatorname{Re} \Phi_{ij} \, d \operatorname{Im} \Phi_{ij} \prod_{i=1}^{N} d\Phi_{ii}.$$

$\mathscr{Z}_{N,M}^2$ is a normalization constant (partition function). We will be interested in the spectral properties of matrices picked randomly with respect to (1.1). The measure (1.1) is invariant under conjugation by unitary matrices, but the matrix elements are not independent unless the polynomial V is quadratic. In the case of quadratic V, we get the Gaussian Unitary Ensemble (GUE), properly scaled. Our results will also apply to the corresponding invariant ensembles of symmetric matrices and so-called self-dual $2N \times 2N$ hermitian matrices. On the set of all $N \times N$ real symmetric matrices $\Phi = (\Phi_{ij})$, we put the probability measure

$$\frac{1}{\mathscr{Z}_{N,M}^1} \exp\left(-\frac{M}{2} \operatorname{Tr} V(\Phi)\right) \prod_{1 \leqslant i \leqslant j \leqslant N} d\Phi_{ij},$$

which is invariant under conjugation by orthogonal matrices. If V is quadratic, the matrix elements are independent, real, normal, random variables, and we get the Gaussian Orthogonal Ensemble (GOE). See [Me] for the definitions of self-dual hermitian matrices and the appropriate measure (symplectic ensembles). For historical background and physical motivation, see [Me] and [Po]. Random matrices are also relevant in connection with quantum chaos; see [GVZ] for reviews.

In the study of the spectral properties of random matrices, most of the interest has centered around the *local* fluctuation properties of the spectrum, for exam-

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