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LINEAR DILATATION OF QUASICONFORMAL MAPS IN SPACE

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1. Introduction. The purpose of this paper is to present an explicit sharp upper bound for the linear dilatation of *K-quasiconformal* (K-qc) maps. This bound yields a solution to a well-known open problem.

Let f be a homeomorphism from a domain D of $\mathbf{R}^n$ into $\mathbf{R}^n$. Then

$$H(x,f) = \limsup_{r\to 0+} \frac{L(x,f,r)}{l(x,f,r)}$$

is the *linear dilatation* of f, where

$$L(x,f,r) = \max_z\{|f(z)-f(x)| : |z-x| = r\},$$

$$l(x,f,r) = \min_z\{|f(z)-f(x)| : |z-x| = r\},$$

$x \in D$, and $\bar{B}(x,r) \subset D$. For domains D of $\mathbf{R}^n$ and K-quasiconformal maps (see Section 2.1), we define

$$H(K,D) = \sup\{H(x,f) : x \in D, f\colon D \to D' \text{ is } K\text{-qc, where } D' \subset \mathbf{R}^n\}.$$

A. Mori [M] proved in 1957 that $H(K,D) \leqslant e^{\pi K}$ when D is a subdomain of $\mathbf{R}^2$. In 1959, O. Lehto, K. I. Virtanen, and J. Väisälä [LVV] found the exact value for $H(K,\mathbf{R}^2)$. In terms of τ_2, the capacity of the Teichmüller ring (see Section 2.1),

$$\lambda(K) \equiv H(K,\mathbf{R}^2) = \tau_2^{-1}(\tau_2(1)/K). \tag{1.1}$$

Dao-shing Shah and Le-le Fan [SF] proved in 1960 that $H(K,D)$ has the same value also when D is a subdomain of $\mathbf{R}^2$. In 1962, F. W. Gehring [G] proved that

$$H(K,D) \leqslant d(n,K) \equiv \exp\left(\left(\frac{K\omega_{n-1}}{\tau_n(1)}\right)^{1/(n-1)}\right), \tag{1.2}$$

where D is a subdomain of $\mathbf{R}^n$ and ω_{n-1} is the surface area of the unit sphere $\mathbf{S}^{n-1}$. Note that for $n = 2$, Gehring's inequality (1.2) gives the bound of Mori, since

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