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ON-DIAGONAL LOWER BOUNDS FOR HEAT KERNELS AND MARKOV CHAINS

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1. Introduction. Let M be a Riemannian manifold, and Δ be the Laplace-Beltrami operator on M. It is known that there exists a unique minimal positive fundamental solution to the associated heat equation, which is referred to as the heat kernel and denoted by $p_t(x, y)$ $(x, y \in M,\ t > 0)$.

For example, in $\mathbb{R}^n$, the heat kernel is given by the explicit formula

$$p_t(x, y) = \frac{1}{(4\pi t)^{n/2}} \exp\left(-\frac{|x - y^2|}{4t}\right),$$

which shows that $p_t(x, y)$ behaves like $t^{-n/2}$ for fixed x and y as $t \to \infty$. On other manifolds, its behaviour may be described by different functions of t, depending on the geometry of the manifold.

The major question on complete noncompact manifolds is: What geometric terms are adequate to describe the long-time behaviour of the heat kernel? Much is known about upper bounds. The seminal works of Nash [N], Aronson [Ar], Varopoulos [V2], Carlen, Kusuoka, and Stroock [CKS], and Davies [D1] brought the understanding that the uniform upper bounds of the heat kernel are closely related to isoperimetric-type inequalities, including the Sobolev, Nash, and logarithmic Sobolev inequalities. More recent works [G2], [Carr], [C2] revealed the importance of a Faber-Krahn–type inequality and of a generalized Nash inequality (see also the surveys [G4] and [C3]).

The situation is quite different with lower bounds of the heat kernel. Until recently, only two methods were known:

- using a comparison-type theorem [DGM], [ChY], which requires a pointwise restriction on the Ricci curvature;
- using an upper bound of the heat kernel, either directly or together with a uniform Harnack inequality or equivalent tools (see [N], [Ar], [LY], [FS], [BCF], [PE]).

The latter approach needs additional ingredients, essentially Poincaré-type inequalities and the doubling volume property (see [S], [G6]). In particular, this means that it can only treat a polynomial decay of the heat kernel and that its natural range is limited to manifolds with a nonnegative Ricci curvature (up to quasi-isometry). On the other hand, both above approaches give at the same time off-diagonal bounds for $p_t(x, y)$.

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