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GEOMETRIC CATEGORIES AND O-MINIMAL STRUCTURES

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Introduction. The theory of subanalytic sets is an excellent tool in various analytic-geometric contexts; see, for example, Bierstone and Milman [1]. Regrettably, certain "nice" sets—such as $\{(x, x^r): x > 0\}$ for positive irrational r, and $\{(x, e^{-1/x}): x > 0\}$—are not subanalytic (at the origin) in $\mathbb{R}^2$. Here we make available an extension of the category of subanalytic sets that has these sets among its objects and that behaves much like the category of subanalytic sets. The possibility of doing this emerged in 1991 when Wilkie [27] proved that the real exponential field is "model complete," followed soon by work of Ressayre, Macintyre, Marker and the authors; see [21], [5], [8], and [19]. However, there are two obstructions to the use by geometers of this development: (i) while the *proofs* in these articles make essential use of model theory, many *results* are also stated there (efficiently, but unnecessarily) in model-theoretic terms; (ii) the results of these papers apply directly only to the cartesian spaces $\mathbb{R}^n$, and not to arbitrary real analytic manifolds. Consequently, in order to carry out our goal, we recast here some results in those papers—as well as many of their consequences—in more familiar terms, with emphasis on results of a geometric nature, and allowing arbitrary (real analytic) manifolds as ambient spaces. We thank W. Schmid and K. Vilonen for their suggestion that this would be a useful undertaking. Indeed, they gave us a "wish list" (inspired by Chapters 8 and 9 of Kashiwara and Schapira [12]; see also §10 of [22]) that strongly influenced the form and content of this paper.

We axiomatize in Section 1 the notion of "behaving like the category of subanalytic sets" by introducing the notion of "analytic-geometric category." (The category $\mathcal{C}_{\mathrm{an}}$ of subanalytic sets is the "smallest" analytic-geometric category.) We also state in Section 1 a number of properties shared by all analytic-geometric categories. Proofs of the more difficult results of this nature, like the Whitney-stratifiability of sets and maps in such a category, often involve the use of charts to reduce to the case of subsets of $\mathbb{R}^n$. For subsets of $\mathbb{R}^n$, there already exists the theory of "o-minimal structures on the real field" (defined in Section 2). This subject is developed in detail in [4] and is an abstraction of the theory of semialgebraic sets (see, e.g., Bochnak et al. [2]). Each analytic-geometric category arises in a natural way from an o-minimal structure on the real field.

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