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BINOMIAL IDEALS

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Introduction. It is notoriously difficult to deduce anything about the structure of an ideal or scheme by directly examining its defining polynomials. A notable exception is that of monomial ideals. Combined with techniques for making flat degenerations of arbitrary ideals into monomial ideals (typically, using Gröbner bases), the theory of monomial ideals becomes a useful tool for studying general ideals. Any monomial ideal defines a scheme whose components are coordinate planes. These objects have provided a useful medium for exchanging information between commutative algebra, algebraic geometry, and combinatorics.

This paper initiates the study of a larger class of ideals whose structure can still be interpreted directly from their generators: binomial ideals. By a *binomial* in a polynomial ring $S = k[x_1, \dots, x_n]$, we mean a polynomial with at most two terms, say $ax^\alpha + bx^\beta$, where $a, b \in k$ and $\alpha, \beta \in \mathbf{Z}_+^n$. We define a *binomial ideal* to be an ideal of S generated by binomials, and a *binomial scheme* (or *binomial variety*, or *binomial algebra*) to be a scheme (or variety or algebra) defined by a binomial ideal. For example, it is well known that the ideal of algebraic relations on a set of monomials is a prime binomial ideal (Corollary 1.3). In Corollary 2.6 we shall see that every binomial prime ideal has essentially this form.

A first hint that there is something special about binomial ideals is given by the following result, a weak form of what is proved below (see Corollary 2.6 and Theorem 6.1).

THEOREM. *The components (isolated and embedded) of any binomial scheme in affine or projective space over an algebraically closed field are rational varieties.*

By contrast, every scheme may be defined by trinomials, that is, polynomials with at most three terms. The trick is to introduce $n-3$ new variables z_i for each equation $a_1x^{m_1} + \cdots + a_nx^{m_n} = 0$ and replace this equation by the system of $n-2$ new equations

$$z_1 + a_1x^{m_1} + a_2x^{m_2} = -z_1 + z_2 + a_3x^{m_3} = -z_2 + z_3 + a_4x^{m_4} = \cdots$$

$$\cdots = -z_{n-4} + z_{n-3} + a_{n-2}x^{m_{n-2}} = -z_{n-3} + a_{n-1}x^{m_{n-1}} + a_nx^{m_n} = 0.$$

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