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SMOOTHING EFFECTS OF SCHRÖDINGER EVOLUTION GROUPS ON RIEMANNIAN MANIFOLDS

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0. Introduction. Let (M, g) be a connected, complete Riemannian manifold, and $\Delta = \Delta_g$ the Laplace-Beltrami operator. Throughout this paper, a manifold means a C^∞ manifold with countable open base. Denote the Friedrichs extension of $\Delta|_{C_0^\infty(M)}$ by the same symbol Δ. The aim of this article is to analyse the relationship between a certain smoothing effect of the Schrödinger evolution group $e^{it\Delta}$ and the behavior of the geodesic flow, especially when there exists a complete geodesic contained in a compact subset.

We now illustrate the problem. First, what kind of smoothing effect are we concerned with?

Example 1. Let (M, g) be the connected, simply connected, complete Riemannian manifold of constant curvature $-\rho^2$, $(\rho \geqslant 0)$: the Euclidean space if $\rho = 0$, and the hyperbolic space if $\rho > 0$. Then, by continuity, the map

$$C_0^\infty(M) \ni u \mapsto e^{it\Delta}u \in L^2_{\mathrm{loc}}(\mathbf{R}_t; H^{1/2}_{\mathrm{loc}}(M))$$

can be extended to the following

$$(*) \qquad L^2(M) \ni u \mapsto e^{it\Delta}u \in L^2_{\mathrm{loc}}(\mathbf{R}_t; H^{1/2}_{\mathrm{loc}}(M)).$$

In general, $H^s_{\mathrm{loc}}(M)$ is the set of all $u \in \mathscr{D}'(M)$ such that $\chi^*(\phi u) \in H^s(\mathbf{R}^d) = \{f \in \mathscr{S}'(\mathbf{R}^d): (1 + |D|^2)^{s/2} f \in L^2(\mathbf{R}^d)\}$ for any local chart (U, χ) and for any $\phi \in C_0^\infty(U)$; it has a natural Fréchet space structure. The continuity of $(*)$ implies that $e^{it\Delta}u \in H^{1/2}_{\mathrm{loc}}(M)$ for almost every $t \in \mathbf{R}$, an expression of smoothing effects of $e^{it\Delta}$. The smoothing effect we shall study is the continuity of $(*)$.

Second, what kind of relationship exists between the smoothing effect above and the behavior of the geodesic flow?

Example 2. Let (M, g) be the same as in Example 1, and let h be another Riemannian metric in M, which is equal to g outside a compact subset. Then the map

$$L^2(M) \ni u \mapsto e^{it\Delta_h}u \in L^2_{\mathrm{loc}}(\mathbf{R}; H^{1/2}_{\mathrm{loc}}(M))$$

is continuous if and only if there is no complete geodesic contained in a compact subset. Thus, the existence of trapped geodesics breaks the smoothing effect.

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