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ON THE LIEB-THIRRING ESTIMATES FOR THE PAULI OPERATOR

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1. Introduction. The aim of this paper is to establish some spectral properties of the Pauli operator, that is, of the operator describing the motion of a particle with spin in a magnetic field. We confine ourselves to the case when the spin is allowed to take one of the values $+1/2$ or $-1/2$. The operator acts in $L^2(\mathbb{R}^d) \otimes \mathbb{C}^2$ with $d = 2$ or $d = 3$ and has the form

$$H_{\text{Pauli}}^{(d)} = H_{\mathbf{a}}^{(d)}\mathbb{I} - \Sigma \cdot \mathbf{B}, \qquad \mathbb{I} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

where $H_{\mathbf{a}}^{(d)} = (-i\nabla - \mathbf{a})^2$ is the usual spinless Schrödinger operator with the magnetic vector-potential $\mathbf{a} = \{a_1, \dots, a_d\}$, $\mathbf{B} = \nabla \times \mathbf{a}$ is the field and Σ stands for the vector $\sigma_1, \sigma_2, \sigma_3$ of 2×2 Pauli matrices (see [4]). Suppose that the field $\mathbf{B}$ is pointed along the x_3-axis, i.e., $\mathbf{a} = (a_1, a_2, 0)$ with $a_k = a_k(x_1, x_2)$ (which is always true for $d = 2$). In this case, $\mathbf{B} = (0, 0, B), B = \partial_1 a_2 - \partial_2 a_1$, and $H_{\text{Pauli}}^{(d)}$ looks especially simple:

$$H_{\text{Pauli}}^{(2)} = \begin{pmatrix} A_+ & 0 \\ 0 & A_- \end{pmatrix}, \qquad A_\pm = H_{\mathbf{a}}^{(2)} \mp B, \tag{1.1}$$

$$H_{\text{Pauli}}^{(3)} = H_{\text{Pauli}}^{(2)} + \begin{pmatrix} -\partial_3^2 & 0 \\ 0 & -\partial_3^2 \end{pmatrix}. \tag{1.2}$$

Although this operator does not seem to be nonnegative, the entries $A_\pm$ can be rewritten as $A_\pm = Q_\pm^* Q_\pm$ with the operators

$$Q_\pm = \Pi_1 \pm i\Pi_2, \qquad \Pi_k = -i\partial_k - a_k, \ k = 1, 2, \tag{1.3}$$

which allows one to define $H_{\text{Pauli}}^{(2)}$ as a nonnegative selfadjoint operator (see Section 2 below). A remarkable property of $H_{\text{Pauli}}^{(2)}$ is that the point $\lambda = 0$ belongs to its spectrum. This assertion was proved under fairly broad conditions on the magnetic field B (see [1], [8] and also [4], [5], [10]). If the operator (1.1) or (1.2) is per-

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