ARC STRUCTURE OF SINGULARITIES

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Introduction. This paper is motivated as much by some interesting possible truths which were encountered as by the things which could be established. Perhaps others will be able to complete the picture with the solutions of these implicit problems.

For simplicity we confine ourselves to the complex numbers. We consider abstract (algebraic) varieties over the complex numbers. One could also extend the results to complex analytic varieties if one had the resolution of singularities in this case.

Arcs are small portions of algebroid curves lying in the variety V. They are represented by convergent power series in a complex parameter t giving the coordinates $x(t)$ of a point of V. We consider the nonsingular arcs where $x(t)$ is nonsingular on V for general t. Associated with any algebraic subset W of the variety, there are the arcs where $x(0)$ is a point of W. From the existence of a resolution of the singularities of V, we find that the set of arcs associated with W decomposes into a finite number of families. The number of these is not more than the number of components of the image of W in a resolution. However, the families exist independently of any particular resolution.

For surfaces it seems possible that there are exactly as many families of arcs associated with a point as there are components of the image of the point in the minimal resolution of the singularities of the surface. This is the first open question.

In higher dimensions, the arc families associated with the singular set correspond to "essential components" which must appear in the image of the singular set in all resolutions. We do not know how complete is the representation of essential components by arc families.

Exemplifying essential components is a large class characterized by a condition on their birational types. A hypersurface appearing in a resolution as a component of the image of the singular set which is not birationally equivalent to a ruled variety is essential. It must appear in any resolution.

As illustrations, we present a few examples of singularities and their arc families and essential components.

Arc families. Let W be an algebraic subset of a variety V, and let $x_1, \ldots, x_N$ be locally valid coordinates corresponding to an affine space A^N. We consider

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