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POLYNOMIAL CONVEXITY, RATIONAL CONVEXITY, AND CURRENTS

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0. Introduction. Given a compact set X in $\mathbb{C}^k$ we consider the polynomially convex hull $\hat{X}$ of X and the rationally convex hull $r(X)$. More precisely, a point $x \in \mathbb{C}^k$ is in $\hat{X}$ if and only if $|f(x)| \leqslant \sup_{z \in X} |f(z)|$ for every holomorphic polynomial f. A point $x \in r(X)$ if and only if every algebraic hypersurface through x intersects X. We want to describe the structure of $\hat{X} \backslash X$ (resp. $r(X) \backslash X$).

Assume $f: D \to \mathbb{C}^k$ is a bounded holomorphic map from the unit disc $D \subset \mathbb{C}$, into $\mathbb{C}^k$. If $f^*(e^{i\theta}) \in X$ for almost every $\theta \in \partial D$, then $f(D) \subset \hat{X}$. This fact suggests that $\hat{X} \backslash X$ has some "analytic structure." In many interesting cases it is possible to construct analytic discs, see [We2]. However this is not always possible. Wermer [We1] has constructed a striking example of a compact set $X \subset \{(z, w), |z| = 1\}$ such that $\hat{X} \backslash X$ has no analytic structure in the above sense.

It is also classical that if S is a compact Riemann surface with boundary such that ∂S bounds a surface $\Sigma \subset X$, then $S \subset r(X)$, see Stolzenberg [St1]. However it is still possible that no such (S, Σ) exists but $r(X) \backslash X$ is nonempty.

A natural idea is to replace analytic sets with boundary on X, by positive closed (1, 1) currents on $\mathbb{C}^k \backslash X$ with bounded support. This explains quite well Wermer's example, but still there are cases where such currents do not exist on $\hat{X} \backslash X$. So for polynomial convexity we have to consider positive currents T of bidimension (1, 1) on $\mathbb{C}^k \backslash X$, with bounded support such that $dd^c T \leqslant 0$ on $\mathbb{C}^k \backslash X$. Quite surprisingly, this gives a complete description of $\hat{X} \backslash X$.

Positive currents play a central role in our approach to rational convexity and polynomial convexity.

We describe more precisely the content of the paper.

In Paragraph 1 we prove the following result. Let $T = dd^c \varphi$ be a positive current of bidimension $(k-1, k-1)$ in $\mathbb{C}^k$. This means that the function φ is plurisubharmonic (p.s.h. for short). We prove that $\mathbb{C}^k \backslash(\text{support } T)$ can be exhausted by rationally convex compact sets. Moreover the current T is the limit of a sequence of currents $(1/N_p)[H_p]$, where $[H_p]$ denotes the current of integration on the hypersurface H_p, and $(H_p \cap B)$ converges in the Hausdorff metric towards $\operatorname{supp} T \cap B$. Without this condition the result is classical. The theorem relies heavily on Hörmander's L^2 estimates which give the technique for constructing hypersurfaces.

In Paragraph 2 we study rational convexity for compact sets. We first give a

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