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ON THE BOUNDARIES OF SPECIAL LAGRANGIAN SUBMANIFOLDS

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0. Introduction. In [HL1], Harvey and Lawson prove that Re dz ($dz = dz_1 \wedge \cdots \wedge dz_n$) is a calibration on $\mathbf{C}^n = \mathbf{R}^{2n}$, where $z = (z_1, \ldots, z_n)$ are the coordinates on $\mathbf{C}^n$. Let P be an n-dimensional oriented plane in $\mathbf{R}^{2n}$ with oriented orthonormal basis $\{e_1, \ldots, e_n\}$. Then $\zeta_P = e_1 \wedge \cdots \wedge e_n \in \wedge^n \mathbf{R}^{2n}$ does not depend on the choice of the basis. We say P is *special Lagrangian* if Re $dz(\zeta_P) = 1$. An n-dimensional oriented submanifold M of $\mathbf{C}^n$ is called special Lagrangian if each of its oriented tangent planes is special Lagrangian, or equivalently, if its volume form is given by the restriction of the ambiant form Re dz. Special Lagrangian submanifolds are volume minimizing. One can show that an oriented submanifold M (with proper oriention) is special Lagrangian if and only if the restrictions to M of ω and Im dz are zero, where ω is the standard Kähler/symplectic form on $\mathbf{C}^n = \mathbf{R}^{2n}$. In particular, special Lagrangian submanifolds are Lagrangian (i.e., the restriction of ω to M is zero). A Lagrangian submanifold is minimal if and only if there exists a real number θ such that its volume form is given by the restriction of Re$\{e^{i\theta}\, dz\}$. Consequently, the study of a minimal Lagrangian submanifold is reduced to the study of a special Lagrangian submanifold. See [HL1] for the details.

On $\mathbf{C}$, special Lagrangian submanifolds are lines parallel to the x-axis and their geometry is trival. Therefore, throughout this paper, we only consider $\mathbf{C}^n$ for $n \geqslant 2$.

To understand the special Lagrangian geometry on $\mathbf{C}^2$, we introduce another copy of $\mathbf{C}^2$, denoted by $\mathscr{C}^2$, and let (w_1, w_2) be its coordinates. Define a diffeomorphism $F\colon \mathbf{C}^2 \to \mathscr{C}^2$ by $F(z_1, z_2) = (w_1, w_2)$, where $w_1 = x_1 + ix_2$, $w_2 = y_2 + iy_1$, and x_k, y_k are determined by $z_k = x_k + iy_k$ ($k = 1, 2$). It is easy to see that under this diffeomorphism Re $dz_1 \wedge dz_2$ corresponds to $(i/2)(dw_1 \wedge d\overline{w}_1 + dw_2 \wedge d\overline{w}_2)$. That is, the special Lagrangian calibration Re dz on $\mathbf{C}^2$ corresponds to the Kähler calibration on $\mathscr{C}^2$. So a submanifold of $\mathbf{C}^2$ is special Lagrangian if and only if it corresponds to a complex submanifold of $\mathscr{C}^2$. Therefore the special Lagrangian geometry on $\mathbf{C}^2$ is the same as the complex geometry on $\mathscr{C}^2$.

Let $\tilde{J}$ be the complex structure on $\mathbf{C}^2$ defined by pulling back the complex structure on $\mathscr{C}^2$, that is

$$\tilde{J}\left(\frac{\partial}{\partial x_1}\right) = \frac{\partial}{\partial x_2}, \quad \tilde{J}\left(\frac{\partial}{\partial x_2}\right) = -\frac{\partial}{\partial x_1}, \quad \tilde{J}\left(\frac{\partial}{\partial y_2}\right) = \frac{\partial}{\partial y_1}, \quad \tilde{J}\left(\frac{\partial}{\partial y_1}\right) = -\frac{\partial}{\partial y_2}.$$

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