Vol. 79, No. 1 DUKE MATHEMATICAL JOURNAL © July 1995

CLASS GROUP L-FUNCTIONS

W. DUKE, J. FRIEDLANDER, AND H. IWANIEC

To Professor Wolfgang Schmidt on the occasion of his sixtieth birthday.

1. Introduction. We consider the imaginary quadratic field $K = \mathbb{Q}(\sqrt{-D})$ of discriminant $-D$. Our main interest here is to study the L-functions of K which are attached to the characters of the class group $\mathscr{H}$. Letting $\chi \in \hat{\mathscr{H}}$ be such a character, we denote

$$L_K(s, \chi) = \sum_{\mathfrak{a}} \chi(\mathfrak{a})(N\mathfrak{a})^{-s}, \tag{1.1}$$

where $\mathfrak{a}$ ranges over nonzero integral ideals, and $N\mathfrak{a}$ is the norm. Thus we have $h = h(-D) = |\mathscr{H}|$ such L-functions. It is known that the "class number" h satisfies

$$D^{1/2-\varepsilon} \ll h \ll D^{1/2} \log D. \tag{1.2}$$

Here the lower bound (ineffective) is due to C. L. Siegel [Si], and the upper bound is elementary.

For the trivial character $\chi = 1$, the L-function is just the Dedekind zeta-function of K and can be expressed as $\zeta_K(s) = \zeta(s)L(s, \chi_{\mathrm{D}})$, where $\zeta(s)$ is the Riemann zeta-function and $L(s, \chi_{\mathrm{D}})$ is the Dirichlet L-function for the field character $\chi_{\mathrm{D}}(n) = (-D/n)$. More generally, if χ is real we have Kronecker's factorization

$$L_K(s, \chi) = L(s, \chi_{D_1})L(s, \chi_{D_2}), \tag{1.3}$$

where $-D = D_1 D_2$ is some factorization into fundamental discriminants $-D_1$ and $-D_2$, see [Si2, pp. 81 and 91]. Of course the number $2^{\omega(D)-1}$ of such real characters is quite small in comparison to h when D is large.

The L-functions defined in (1.1) for $\operatorname{Re} s > 1$, where the series converges absolutely, possess an analytic continuation to the whole complex s-plane. They are entire functions except for $\zeta_K(s)$, which has a simple pole at $s = 1$ of residue

$$\operatorname*{res}_{s=1} \zeta_K(s) = L(1, \chi_{\mathrm{D}}) = 2\pi h w^{-1} D^{-1/2}, \tag{1.4}$$

Received 17 June 1994.
Duke and Iwaniec supported by NSF grant DMS-9202022.
Friedlander and Iwaniec supported by NSERC grant A5123.