Vol. 78, No. 2 DUKE MATHEMATICAL JOURNAL © May 1995

KOENIGS FUNCTIONS, QUASICIRCLES AND BMO

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1. Introduction. Let $R: \hat{\mathbf{C}} \to \hat{\mathbf{C}}$ be a rational function having an attracting fixpoint $z_0 \in \mathbf{C}$ and consider its immediate basin of attraction G, that is, the component of the Fatou set containing z_0. We assume that the multiplier $\lambda = R'(z_0)$ satisfies $0 < |\lambda| < 1$. The Koenigs function f_R of R is the analytic function which conjugates R to its linear approximation,

$$f_R \circ R(z) = \lambda f_R(z), \tag{1.1}$$

normalized by $f_R'(z_0) = 1$. Note that $f_R(z_0) = 0$. Since R is rational, f_R is analytic in all of G, and not only in a neighborhood of z_0. We refer to [B, Chapter 6.3], [CG, Chapter II], or [S, Chapter 3.4] for the details of this discussion.

There is another natural conjugacy. If G is simply connected and if $\phi: \mathbf{D} \to G$ is a conformal map from the unit disk $\mathbf{D}$ onto G, then

$$B = B_R = \phi^{-1} \circ R \circ \phi \tag{1.2}$$

is an analytic proper self-map of the unit disk, and thus a Blaschke product. If G is multiply connected, R lifts to an inner function B of the unit disk via a covering map $\phi: \mathbf{D} \to G$, $\phi(0) = z_0$. Therefore, in any case, the analytic function $f_B = f_R \circ \phi$ is the Koenigs function of B, conjugating B to a linear map near the origin.

Since $f_R'(z_0) = 1$, there is some, and hence the largest, disk D_R centered at 0 such that f_R^{-1} can be defined and is analytic in D_R. Thus

$$C_R = f_R^{-1}(D_R) \subset G$$

is the largest subdomain of G that contains z_0 and is mapped univalently by f_R onto some disk centered at the origin. Similarly, $C_B = f_B^{-1}(D_R) = \phi^{-1}(C_R)$ is the largest disk that contains 0 and is mapped univalently by f_B onto some disk centered at the origin.

Recall that a K-quasidisk is the image of a disk under a K-quasiconformal self-map of the sphere $\hat{\mathbf{C}}$. A boundary of a K-quasidisk is termed a K-quasicircle. In this paper we shall establish a new criterion for a domain to be a quasidisk (Theorem 1.4), and our main application is the following description of C_B.

Received 21 February 1994.

Heinonen supported by the National Science Foundation, the Academy of Finland, and the A. P. Sloan Foundation.

Rohde supported by the Alexander von Humboldt-Stiftung.