Vol. 77, No. 3 DUKE MATHEMATICAL JOURNAL © March 1995

PERTURBATIONS OF THE ROTATION C^*-ALGEBRAS AND OF THE HEISENBERG COMMUTATION RELATION

UFFE HAAGERUP AND MIKAEL RØRDAM

1. Introduction. It is proved that almost commuting operators on a Hilbert space in specific cases of interest are close to commuting operators if the given operators are amplified infinitely.

Let P and Q be (unbounded) selfadjoint operators on a Hilbert space H satisfying the Heisenberg commutation relation $PQ - QP = -iI$, and let K be an infinite-dimensional Hilbert space. We show (Theorem 3.1) that there are commuting selfadjoint operators P_0 and Q_0 on $H \otimes K$ such that $P \otimes I - P_0$ and $Q \otimes I - Q_0$ are bounded.

Let S and Ω be the Voiculescu matrices in $U(n)$ which satisfy $S\Omega = \omega\Omega S$ where $\omega = \exp(2\pi i/n)$ (see Corollary 4.12). Let H be an infinite-dimensional Hilbert space. It is proved that there are commuting unitaries S_0 and Ω_0 on $\mathbb{C}^n \otimes H$ so that $\|S \otimes I - S_0\|$ and $\|\Omega \otimes I - \Omega_0\|$ are less than $25n^{-1/2}$.

The rotation C^*-algebra A_θ, $\theta \in \mathbb{R}$, associated with the rotation of the circle by angle $2\pi\theta$, is the universal C^*-algebra generated by two unitaries u and v satisfying the commutation relation

$$uv = e^{2\pi i\theta} vu.$$

G. Elliott has in [8] proved that the family of rotation C^*-algebras forms a continuous field in the sense that there is a C^*-algebra $\mathscr{A}$ and surjective $*$-homomorphisms $\pi_\theta \colon \mathscr{A} \to A_\theta$ such that the maps $\theta \mapsto \|\pi_\theta(a)\|$ are continuous for all $a \in \mathscr{A}$. We prove that the rotation C^*-algebras form a continuous field in the following stronger sense.

Let H be an infinite-dimensional separable Hilbert space. Then there exist two continuous paths $u, v\colon [0, 1] \to U(H)$ into the unitary group $U(H)$ of H such that $u(0) = u(1)$, $v(0) = v(1)$, and $u(\theta)v(\theta) = \exp(2\pi i\theta)v(\theta)u(\theta)$ for each $\theta \in [0, 1]$. Moreover, u, v can be chosen such that

$$\max\{\|u(\theta_2) - u(\theta_1)\|, \|v(\theta_2) - v(\theta_1)\|\} \leq C|\theta_1 - \theta_2|^{1/2}$$

for all $\theta_1, \theta_2 \in [0, 1]$ and where C is a universal constant (see Theorem 5.4). This estimate is (up to a factor) best possible in the sense that we also have

$$\max\{\|u_2 - u_1\|, \|v_2 - v_1\|\} \geqslant |\theta_1 - \theta_2|^{1/2},$$

Received 25 April 1994.