Vol. 77, No. 3 DUKE MATHEMATICAL JOURNAL © March 1995

HODGE CLASSES AND TATE CLASSES ON SIMPLE ABELIAN FOURFOLDS

B. J. J. MOONEN AND YU. G. ZARHIN

1. Introduction. Let X be a smooth projective variety over the field $\mathbb{C}$ of complex numbers. As is well known, the nth cohomology group $\mathrm{H}^n(X, \mathbb{Q})$ carries a Hodge structure of weight n. This gives us a decomposition

$$\mathrm{H}^n(X, \mathbb{C}) = \bigoplus_{p+q=n} \mathrm{H}^{p,q},$$

such that the subspaces $\mathrm{H}^{p,q}$ and $\mathrm{H}^{q,p}$ are complex conjugate to each other.

A rational cohomology class $c \in \mathrm{H}^{2p}(X, \mathbb{Q})$ is called a *Hodge class* if it is purely of type (p, p), i.e., if $c \in \mathrm{H}^{2p}(X, \mathbb{Q}) \cap \mathrm{H}^{p,p}$. It is known that

(a) any algebraic cohomology class (i.e., the cohomology class of an algebraic cycle on X) is a Hodge class and

(b) any Hodge class in $\mathrm{H}^2(X, \mathbb{Q})$ is a rational multiple of a divisor class (the Lefschetz theorem on (1, 1)-classes).

The famous and still unproven Hodge Conjecture asserts that

(c) any Hodge class is a rational multiple of an algebraic cohomology class.

A cohomology class c is called *decomposable* if it is an element of the $\mathbb{Q}$-subalgebra of $\bigoplus_{n \geq 0} \mathrm{H}^n(X, \mathbb{Q})$ generated by divisor classes, or, stated differently, if it can be expressed as a linear combination with rational coefficients of cup-products of divisor classes. Since cup-product is compatible with the Hodge decomposition (in the obvious sense), it follows from (b) that

(d) if all Hodge classes on X are decomposable then the Hodge conjecture is true for X. For example, it follows from (b) and the Hard Lefschetz Theorem, that all Hodge classes on a variety X with $\dim(X) < 4$ are decomposable, hence the Hodge conjecture is true for such varieties.

From now on we will assume that X is an abelian variety. Let us write $V = \mathrm{H}_1(X, \mathbb{Q})$ for its first rational homology group, then there is a natural isomorphism of Hodge structures

$$\mathrm{H}^n(X, \mathbb{Q}) \cong \mathrm{Hom}\left(\bigwedge^n V, \mathbb{Q}\right).$$

Under this isomorphism cup-product corresponds to the exterior product of alternating forms.

Received 18 January 1994. Revision received 25 August 1994.
Supported by the Netherlands Organization for Scientific Research (N.W.O.).