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ON SHIFTED CONVOLUTIONS OF $\zeta^3(s)$ WITH AUTOMORPHIC L-FUNCTIONS

NIGEL J. E. PITT

Introduction. Let $f(z)$ be a holomorphic cusp form of weight κ for the modular group $SL_2(\mathbb{Z})$. If we write its Fourier expansion as

$$f(z) = \sum_{n=1}^{\infty} a(n) n^{(\kappa-1)/2} e(nz)$$

where $e(\xi) = e^{2\pi i \xi}$, then the size of the Fourier coefficients is controlled by Deligne's bound (see [5])

$$a(n) \ll \tau(n).$$

Here $\tau(n)$ denotes the number of positive divisors of n and is bounded by $O(n^{\varepsilon})$ for any $\varepsilon > 0$. The sign of the coefficients $a(n)$ is highly oscillatory as n increases, and leads to cancellation in convolution-type sums

$$\sum_{n \leqslant x} a(n)\bar{b}(n)$$

for various sequences $b(n)$. For instance,

$$\sum_{n \leqslant x} a(n)e(\alpha n) \ll x^{1/2} \log x, \qquad \sum_{n \leqslant x} a(n) \ll \frac{x^{1/3}}{(\log x)^{\delta}}, \qquad (0.1), (0.2)$$

the first being due to Hardy and Ramanujan, uniformly in $\alpha \in \mathbb{R}$, and the second to Rankin [15], for $\delta \sim 1/50$.

If the sequence $b(n)$ are the coefficients of a modular form, then Selberg [16] showed

$$\sum_{n \leqslant x} a(n)\bar{b}(n) = cx + O(x^{(3/5)+\varepsilon}), \qquad (0.3)$$

where the constant c is zero if the form is orthogonal to f. The estimates (0.2), (0.3), as many other sums of this type, are derived from the analytic properties of their associated Dirichlet series

$$\sum_{n=1}^{\infty} a(n)n^{-s}, \qquad \sum_{n=1}^{\infty} a(n)\bar{b}(n)n^{-s}.$$

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