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DIFFERENTIAL OPERATORS, HOLOMORPHIC PROJECTION, AND SINGULAR FORMS

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Introduction. In this paper we present two types of results as applications of the theory of differential operators on hermitian symmetric spaces which we developed in our previous papers. The first subject concerns a projection map which associates a holomorphic modular form h to a nearly holomorphic form f so that $\langle \varphi, f \rangle = \langle \varphi, h \rangle$ holds for every holomorphic cusp form φ, where $\langle\ ,\ \rangle$ is an inner product. In our previous investigations, [S6] for example, we employed the map $f \mapsto h$ in the Hilbert modular case as an indispensable tool for the algebraicity of the critical values of certain zeta functions. In the present paper we extend it to the case of an arbitrary classical group G acting on a hermitian symmetric space $\mathscr{H}$ of noncompact type with future applications to the algebraicity problems for such a G in view.

The second topic is a generalization of the following fact:

(1) *A Siegel modular form is singular in the sense that it is annihilated by certain differential operators if and only if it is of singular weight.*

This was proved by Resnikoff, Maass, and Freitag for scalar-valued forms and later by Howe for vector-valued forms with L^2-integrability. See [R1], [R2], [M], [F1], and [H]. All these concern the forms on $Sp(n, \mathbf{Q})$ except that Resnikoff treated some forms of a certain special type on tube domains. Also [F2] may be mentioned as a recent article for $Sp(n, \mathbf{Q})$ which lists practically all relevant papers. In the present paper we give a uniform treatment applicable to both tube and nontube classical domains and even to the cocompact case.

The main idea behind these two types of results is the relations

$$(2) \qquad \langle D_\rho^Z f, g \rangle = (-1)^p \langle f, \theta E^Z g \rangle, \qquad \langle E^Z f, h \rangle = (-1)^p \langle f, \theta D_{\rho\otimes\sigma_Z}^Z h \rangle.$$

Here Z is an irreducible subspace of the pth symmetric product of the holomorphic tangent space of $\mathscr{H}$, ρ is a representation of the maximal compact subgroup of G, D_ρ^Z, $D_{\rho\otimes\sigma_Z}^Z$, and E^Z are certain differential operators on $\mathscr{H}$ depending on Z and ρ, and θ is a contraction operator. In the simplest case in which $\mathscr{H}$ is the upper half plane and $p = 1$, the operators are $y^{-k}(\partial/\partial z)y^k$ and $y^2\partial/\partial\bar{z}$, where $y = \mathrm{Im}(z)$. It may be noted that both $\theta E^Z D_\rho^Z$ and $\theta D_{\rho\otimes\sigma_Z}^Z E^Z$ are essentially self-adjoint G-covariant operators, which generalize the Casimir operator.

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