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ON A QUESTION OF B. MAZUR

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Let Y be a compact Riemann surface of genus $\geqslant 2$ and T any complex torus. Let $\varphi\colon Y \to T$ be an analytic embedding and let $\Gamma \subset T$ be a subgroup of finite rank (not necessarily finitely generated; e.g., take Γ to be the division point saturation of a finitely generated subgroup of T). A conjecture of Lang [L] (which is now a theorem due to work of Faltings [FW] and Raynaud [R]) says that the intersection $\varphi(Y) \cap \Gamma$ is always finite. In [M] Mazur asked the following question: "Is it reasonable to hope that the cardinality of $\varphi(Y) \cap \Gamma$ admits an upper bound that depends only on the genus of Y and the rank of Γ?" The aim of this note is to show that *such a bound exists indeed, at least if we restrict ourselves to the case when Y is not a covering of the Riemann sphere unramified outside 3 points.* Actually we can do slightly better, by allowing φ to be generically injective, rather than an embedding; this covers the interesting case of singular curves in tori. Note also that our bound will be quite explicit (although huge).

We stated our result in the language of complex analytic geometry only to impress the reader. For what we have here entirely belongs to algebraic geometry: Y is, by Riemann, an algebraic curve, and replacing T by the torus A generated by $\varphi(Y)$ we may assume $T = A$ is an abelian variety. Moreover, by Belyi's theorem [Be], the condition "Y is not a covering of the Riemann sphere unramified outside 3 points" is equivalent to "Y, viewed as an algebraic curve, does not descend to the field $\overline{\mathbf{Q}}$ of algebraic numbers". So we actually prove here that Mazur's question has a positive answer in the "non-arithmetic case".

Note that in our paper [B] we bounded $\sharp(\varphi(Y) \cap \Gamma)$ by a constant depending on the rank of Γ and the degree of $\varphi(Y)$ in some fixed projective embedding of A. Such a result does not answer in principle Mazur's question because A may contain infinitely many curves of a given genus, whose projective degrees go to infinity. Moreover, according to our philosophy in [B], if we set $X = \varphi(Y)$ then the intersection $X \cap \Gamma$ is related to an intersection $X^p \cap S^p$ of two subvarieties of some bigger variety A^p; the degree of X^p (in some compactification of A^p) effectively depends on the projective degree of X, while the degree of S^p depends on the rank of Γ. So in principle the method in [B] gives "via Bézout" a bound for $X \cap \Gamma$ which should inherently depend on the projective degree of X. It is therefore surprising to see, as we shall, that the method in [B] may be successfully applied to Mazur's question; cf. also the remarks made in the course of the proof below.

Here is our main result.

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