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THE GEOMETRY OF DEGREE-FOUR CHARACTERISTIC CLASSES AND OF LINE BUNDLES ON LOOP SPACES I

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1. Introduction. Recent developments in quantum field theory have shed new light on the theory of characteristic classes. Rich geometric structures, which are unexpected from the viewpoint of classical topology, have been revealed. The key result is the relationship between Chern-Simons theory and conformal field theory uncovered by Witten [34]. This is a consequence of a geometric reciprocity law for loop groups [29], [30], [35], which has recently been interpreted arithmetically by Deligne [13]; for a compact Lie group G, those extensions of the loop group LG which lie in the image of the natural map $\tau: H^4(BG) \to H^3(BLG)$ satisfy the reciprocity property. Such extensions give rise to a modular functor [29]. The map τ then gives the relationship with 3-dimensional Chern-Simons theories which are classified by $H^4(BG)$ [15]. These various levels of related geometric structures are all associated to a characteristic class α in $H^4(BG)$, and one of the goals of this paper and its sequel is to account for their origin. We do this by constructing a "geometric object" corresponding to α. This is an example of what L. Breen calls a 2-gerbe [4], [5], and its relationship to α is analogous to that which exists between the first Chern class of a line bundle and the line bundle itself.

We begin in §2 by generalizing the theory of line bundles with connection due to Weil [31] and Kostant [24]. Let ν be a closed, integral 3-form on a 2-connected manifold M. Using the path-loop fibration $PM \to M$, we construct a geometric object $\underline{\mathscr{C}}(\nu)$ on M, for which the 3-form ν plays the role of curvature in a natural way. In fact, we show in Proposition 3.1 that $\underline{\mathscr{C}}(\nu)$ is a sheaf of groupoids and is an example of a structure known in algebraic geometry as a *gerbe* [19]. To make the presentation as self-contained as possible, the theory of gerbes and their classification by degree-2 sheaf cohomology is reviewed briefly in §3.

As $\underline{\mathscr{C}}(\nu)$ comes equipped with a notion of connection, it is possible to consider the holonomy of $\underline{\mathscr{C}}$ around any loop in M. This gives a geometric construction of the "anomaly line bundle" over the free loop space LM corresponding to ν (Theorem 4.4). When the manifold in question is a Lie group G, we obtain a description of the central extension of LG associated to ν (Theorems 5.4 and 5.12). This leads to a geometric proof of the reciprocity law of Segal-Witten in the case where G is simply connected, and a more abstract proof for the general case (Theorem 5.9).

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