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2-CATEGORIES AND 2-KNOTS

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1. Introduction. Following Vaughan Jones's ground-breaking discovery of a new polynomial invariant of links in 1984 [Jon], there gradually emerged a clear picture unifying said polynomial and subsequent generalizations. Building on Joyal-Street's study of braiding in monoidal categories [JS], Freyd-Yetter [FY] —using a description of *tangles* in [Y1]—and Turaev [T1] independently gave presentations of the monoidal category they form. Certain morphisms in such categories describe links. This resulted in the observation that the data required to construct an invariant of links in the three-sphere is contained in any *rigid, braided, monoidal category* which satisfies one additional relation. Such an object successfully algebraicizes the Reidemeister moves [Rei] of classical knot theory.

At about the same time, a large class of such categories was emerging from the representation theory of *quantum groups* [D], which provides solutions to the so-called *Yang-Baxter*, or *triangle*, equation

$$R_{12}R_{13}R_{23} = R_{23}R_{13}R_{12}.$$

Some of these solutions were first exploited by Turaev in [T1] to construct invariants directly from tangles, avoiding the consideration of Markov classes of braids needed in earlier works ([Res], [T2], and, before quantum groups, [Jon]). The topological significance of this equation arises from the fact that solutions give rise to representations of Artin's braid groups—the triangle equality ensures group relations such as

$$\sigma_{12}\sigma_{23}\sigma_{12} = \sigma_{23}\sigma_{12}\sigma_{23}.$$

This captures the notion that, subject to certain crossing data, in Figure 1, the two projections in the plane are equivalent. In statistical mechanics, this geometric relation is shorthand for an equality of two sums over internal edges [Ba1].

The current work is motivated by two main sources. First, observe that any set of Reidemeister-type moves for *knotted surfaces* in *four-space* would include in an essential manner the following situation: suppose the set of double points of the projection of a knotted surface from four-space into three-space contains the edges of a tetrahedron. Then passing a patch of the surface containing one of the faces of the tetrahedron through an opposite triple point should not change the isotopy

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