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A COMPUTATION OF GREEN FUNCTIONS FOR THE WEIGHTED BIHARMONIC OPERATORS $\Delta|z|^{-2\alpha}\Delta$, WITH $\alpha > -1$

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0. Introduction. Let Ω be a simply connected bounded domain in the complex plane $\mathbb{C}$, with smooth boundary $\partial\Omega$. The Green function for the Laplace operator Δ is the solution $\Gamma(z, \zeta; \Omega)$, with parameter values $z, \zeta \in \bar{\Omega}$, to the Poisson equation

$$\begin{cases} \Delta\Gamma(\cdot, \zeta; \Omega) = \delta_\zeta & \text{on } \Omega, \\ \Gamma(z, \zeta; \Omega) = 0, & z \in \partial\Omega, \end{cases}$$

where δ_ζ denotes the unit point mass at the interior point $\zeta \in \Omega$; it is well known and easily checked that this function is symmetric in its arguments: $\Gamma(z, \zeta; \Omega) = \Gamma(\zeta, z; \Omega)$. A fundamental fact in the potential theory of the region Ω is the fact that the Green function has constant sign: $\Gamma(z, \zeta; \Omega) < 0$ on $\Omega^2 = \Omega \times \Omega$. This has the physical interpretation that a membrane always follows the direction of the force, no matter where it is applied. In his 1908 memorial [9, pages 541–543], [10, pages 1298–1299], Jacques Hadamard mentions a conjecture, which he ascribes to Tommaso Boggio [4], stating that the Green function $U(z, \zeta; \Omega)$ for the squared operator Δ^2, which solves

$$\begin{cases} \Delta^2 U(\cdot, \zeta; \Omega) = \delta_\zeta & \text{on } \Omega, \\ U(z, \zeta; \Omega) = 0, & z \in \partial\Omega, \\ \nabla_z U(z, \zeta; \Omega) = 0, & z \in \partial\Omega, \end{cases}$$

where ∇_z denotes the gradient taken with respect to the z variable, should also have constant sign, in this case positive, throughout Ω^2. Hadamard also adds the comment that he considers this very likely for convex regions Ω. That it is so if Ω is the unit disk $\mathbb{D}$ was well known before Hadamard wrote his paper, although I do not really know who first noticed this fact. Still, I should like to point out the 1901 papers [4], [19] by Boggio and John Henry Michell as possible sources. It deserves to be mentioned that there is an 1862 book on elasticity theory by Alfred Clebsch, and papers very close to those of Boggio and Michell by Emilio Almansi (1896) and Giuseppe Lauricella (1896). The solution for the disk has the explicit

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