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THE SMOOTH CLASSIFICATION OF ELLIPTIC SURFACES WITH $b^+ > 1$

ANDRÁS STIPSICZ AND ZOLTÁN SZABÓ

1. Introduction. The classification scheme of elliptic surfaces up to deformation equivalence is a well-understood topic in complex geometry. To begin, we remind the reader of the definition of elliptic surfaces:

Definition 1.1. A compact complex surface S is an elliptic surface if there is a holomorphic map $\pi: S \to C$ from S to a complex curve C such that the generic fiber is a smooth elliptic curve.

It is known that if the genus $g(C)$ is positive, then the deformation type of S is determined by its homotopy type. If $C = \mathbb{CP}^1$, then a family $\mathscr{F} = \{E(n)_{p_1,\ldots,p_r} | n, r, p_1, \ldots, p_r \in \mathbb{N}\}$ can be constructed with the property that every minimal elliptic surface S (fibered over $\mathbb{CP}^1$) is deformation equivalent to an element of $\mathscr{F}$. The surfaces in $\mathscr{F}$ can be built using a basic example and two operations.

The basic example is obtained from a pencil of two cubic curves in general position on $\mathbb{CP}^2$ by blowing up the nine base points. In this way, $\mathbb{CP}^2 \# 9\overline{\mathbb{CP}^2}$ becomes an elliptic surface (see [G]). The first operation mentioned above, the "fiber sum" operation, produces a new elliptic surface from a given pair of elliptic surfaces S_1, S_2. It is defined as follows. Remove a tubular neighborhood of a generic fiber from each and glue together the boundary T^3's using an orientation-reversing, fiber-preserving diffeomorphism φ. The resulting smooth manifold, which is denoted by $S_1 \#_f S_2$, is independent of φ once S_i contains cusp fiber. Denoting our elliptic surface $\mathbb{C}P^2 \# 9\overline{\mathbb{C}P^2}$ by $E(1)$ (although the elliptic structure depends on the choice of cubics, the smooth structure does not), we define $E(n)$ inductively by

$$E(n) = E(n-1) \#_f E(1).$$

The second operation is the logarithmic transformation. Given an elliptic surface $\pi: S \to \mathbb{CP}^1$, delete a tubular neighborhood of a regular (torus) fiber, and glue it back via a diffeomorphism $\phi: \partial(T^2 \times D^2) \to \partial(T^2 \times D^2)$ of the boundary T^3 (note that ϕ is not necessarily fiber-preserving). The multiplicity of ϕ is the absolute value of the degree of $\pi \circ \phi: \{pt.\} \times \partial D^2 \to \partial D^2$. If S has a cusp fiber, and ϕ and ϕ' have the same multiplicity then the resulting surfaces S_ϕ and $S_{\phi'}$ will be diffeomorphic. Applying the logarithmic transformation r times on the space $E(n)$ with

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