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HYPERGEOMETRIC FUNCTIONS AND RINGS GENERATED BY MONOMIALS

ALAN ADOLPHSON

1. Introduction. Let $d^{(1)}, \ldots, d^{(N)} \in \mathbf{Z}^n$ span $\mathbf{R}^n$ as real vector space, and let $a \in \mathbf{C}^n$. To this data, Gelfand, Zelevinskii, and Kapranov [6] associate a system of partial differential equations in variables $\lambda_1, \ldots, \lambda_N$. (See equations (2.4) and (2.5) below.) Let Δ be the convex hull of the points $d^{(1)}, \ldots, d^{(N)}$ and the origin. When there exist integers $b_1, \ldots, b_n$ such that, for $j = 1, \ldots, N$,

$$\sum_{i=1}^{n} b_i d_i^{(j)} = 1 \tag{1.1}$$

(where $d^{(j)} = (d_1^{(j)}, \ldots, d_n^{(j)})$), they prove that the corresponding (left) $\mathscr{D}$-module over the Weyl algebra $\mathscr{D}$ is holonomic and that the space of holomorphic solutions of the system (2.4), (2.5) at a generic point has dimension $n!\,\mathrm{Vol}(\Delta)$, where $\mathrm{Vol}(\Delta)$ denotes the volume of Δ with respect to ordinary Lebesgue measure on $\mathbf{R}^n$. (In [6] it is also assumed that $d^{(1)}, \ldots, d^{(N)}$ generate $\mathbf{Z}^n$ as an abelian group, but this restriction is not important.) Condition (1.1) includes the classical nonconfluent hypergeometric functions (i.e., those with all singularities regular) but not the confluent ones (i.e., those with irregular singularities). The main purpose of this article is to eliminate condition (1.1) and thus include the confluent hypergeometric functions in the theory as well.

We prove that the corresponding $\mathscr{D}$-module is holonomic with no restriction on $d^{(1)}, \ldots, d^{(N)}$ (Theorem 3.9). The problem of showing that the space of holomorphic solutions at a generic point has dimension $n!\,\mathrm{Vol}(\Delta)$ seems more subtle, however. When $a \in \mathbf{C}^n$ is semi-nonresonant, we prove this with no restriction on $d^{(1)}, \ldots, d^{(N)}$ (Corollary 5.20). For general $a \in \mathbf{C}^n$, we have to assume that certain subrings of the ring $R = \mathbf{C}[x^{d^{(1)}}, \ldots, x^{d^{(N)}}]$ (where $x^{d^{(j)}} = x_1^{d_1^{(j)}} \cdots x_n^{d_n^{(j)}}$) are Cohen-Macaulay (Corollary 5.21). Hochster [10] points out that condition (1.1) does not imply that R is Cohen-Macaulay, and hence [6, Theorem 2] is not a special case of our work. However, we believe there is an error in the proof of [6, Theorem 2]. Specifically, in [6, Section 2.4], it is claimed that a certain set of ring elements form a regular sequence. We believe this is not true without further restrictions on $d^{(1)}, \ldots, d^{(N)}$. We make a comment on this difficutly in Section 7. Our Corollaries 5.20 and 5.21 imply that the conclusion of [6, Theorem 2] is true if one assumes additionally either that a is semi-nonresonant or that R is Cohen-Maculay.

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