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ON THE COHOMOLOGY OF KOTTWITZ'S ARITHMETIC VARIETIES

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1. Introduction and notation.

1.1. In this paper we study the cohomology of certain arithmetic subgroups Γ of unitary groups $U(p, q)$, or equivalently, the cohomology of the associated locally symmetric spaces:

$$\Gamma\backslash X_{pq} = \Gamma\backslash U(p, q)/U(p) \times U(q).$$

This question has been studied by many authors, amongst others Kazhdan, Shimura, Wallach, Rapoport, Zink, Rogawski, and recently J.-S. Li. In general, Matsushima's theory, interpreted as in Borel-Wallach [5] and completed by the description of cohomological representations of $U(p, q)$—cf. [34]—provides *a priori* vanishing results on $H^{\cdot}(\Gamma\backslash X_{pq})$ for arbitrary Γ[1]. It is likely that these results cannot be improved for arbitrary Γ; the best theorems along these lines have been obtained by Li [22].

On the other hand, this does not imply that all the possible degrees predicted by the Vogan-Zuckerman theory will actually occur for a given group Γ. This may fail for two reasons. First, it may fail for reasons related to ramification: the group Γ may not be deep enough that the potential cohomology already occurs at its level; for example, there are no modular forms of weight 2 for $SL(2, \mathbb{Z})$. More interestingly, however, there may be groups Γ such that $H^i(\Gamma'\backslash X_{pq}, \mathbb{C}) = 0$ for any subgroup $\Gamma' \subset \Gamma$ of finite index while i is one of the degrees allowed by the cohomological representations of $U(p, q)$.

Such an example has been known for some time: it is provided by the cocompact arithmetic subgroups Γ of $U(1, 2)$ arising from unitary groups over $\mathbb{Q}$ associated to involutions of the second kind on division algebras over quadratic imaginary fields (cf. §1.2 for the construction of these groups). In that case $\Gamma\backslash X_{12}$ is then an arithmetic surface, a quotient of the complex 2-ball X_{12} by an arithmetic group of automorphisms. Rapoport and Zink [27] discovered the phenomenon which is the subject of this paper and proved that $H^1(\Gamma\backslash X_{12}, \mathbb{C}) = \{0\}$ under a restriction on the ramification of Γ at a certain finite place. Langlands [20] suggested using the functoriality correspondence with the quasi-split form of $U(3)$ to re-prove this result by means of representation theory. The general case for $U(3)$ was obtained by

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[1] In fact, the vanishing theorem proper—in this case, $H^i_{var}(\Gamma\backslash X_{pq}) = \{0\}$, $0 < i < \mathrm{Inf}(p, q)$, with the notation of §3.2—was obtained earlier by Borel and Zuckerman.