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SEQUENCES OF METRICS ON COMPACT RIEMANN SURFACES

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1. Definitions and statement of results. A compact Riemann surface S of genus $g \geqslant 2$ can be given canonical Hermitian metrics in three ways: by uniformization, by embedding the surface in its Jacobian, and by mapping the surface into projective space.

Letting $\mathscr{H}$ denote the complex upper half plane, the uniformization theorem says $S \simeq \Gamma\backslash\mathscr{H}$, where Γ is some freely acting Fuchsian subgroup of $PSL(2, \mathbf{R})$. Since the Poincaré metric $ds_P^2 = (dz \otimes d\bar{z})/y^2$ on $\mathscr{H}$ is invariant under the action of $PSL(2, \mathbf{R})$, ds_P^2 induces a metric on S as well. While Γ is determined only up to conjugacy in $PSL(2, \mathbf{R})$, the metric on S is independent of this choice.

The other metrics to be considered are obtained through differential forms on the surface. Using the inner product $\langle \omega, \tau \rangle = (i/2) \int_S \omega \wedge \bar{\tau}$, let $\{\omega_1, \omega_2, \ldots, \omega_g\}$ be any orthonormal basis for the holomorphic differential 1-forms. In $\mathscr{H}$-coordinates $\omega_i = f_i(z)\, dz$ for some holomorphic functions f_i.

Letting $\Lambda = \{\int_\gamma (\omega_1, \omega_2, \ldots, \omega_g) | \gamma \in H_1(S)\} \subseteq \mathbf{C}^g$ denote the period lattice for S, the analytic Jacobian of S is $J(S) = \mathbf{C}^g/\Lambda$. For any fixed $p_0 \in S$, the map $p \mapsto \int_{p_0}^p (\omega_1, \omega_2, \ldots, \omega_g) \bmod \Lambda$ of $S \to J(S)$ is an embedding. The Euclidean metric $\sum_{i=1}^g dz_i \otimes d\bar{z}_i$ on $\mathbf{C}^g$ induces a metric on $J(S)$ which can be pulled back to S to give

$$ds_J^2 = \sum_{i=1}^g \omega_i \otimes \overline{\omega_i} = \sum_{i=1}^g |f_i(z)|^2\, dz \otimes d\bar{z}.$$

While the basis $\{\omega_i\}_{i=1}^g$ was determined only up to unitary transformation, ds_J^2 is independent of that choice and of the choice of p_0.

The last metric to be considered arises from the canonical mapping of S into complex projective space $\mathbf{P}^{g-1}$ by $p \mapsto (f_1(p), f_2(p), \ldots, f_g(p))$. Of course this is not necessarily an embedding since S may be hyperelliptic. Denoting homogeneous coordinates in $\mathbf{P}^n$ by $(Z_0, Z_1, \ldots, Z_n)$, let $t_i = Z_i/Z_0$ denote affine coordinates on the set where $Z_0 \neq 0$. Then the Fubini-Study metric, which is a globally defined, is locally given by

$$ds_{F-S}^2 = \frac{(1 + \sum_{i=1}^n t_i\bar{t_i})(\sum_{i=1}^n dt_i \otimes d\bar{t_i}) - (\sum_{i=1}^n \bar{t_i}\, dt_i) \otimes (\sum_{i=1}^n t_i\, d\bar{t_i})}{(1 + \sum_{i=1}^n t_i\bar{t_i})^2}.$$

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