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ON THE SELBERG CLASS OF DIRICHLET SERIES: SMALL DEGREES

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1. Introduction. In the study of Dirichlet series with arithmetic significance, there has appeared (through the study of known examples) certain expectations, namely (i) if a functional equation and Euler product exists, then it is likely that a type of Riemann hypothesis will hold, (ii) if, in addition, the function has a simple pole at the point $s = 1$, then it must be a product of the Riemann zeta-function and another Dirichlet series with similar properties, and (iii) a type of converse theorem holds, namely that all such Dirichlet series can be obtained by considering Mellin transforms of automorphic forms associated with arithmetic groups. Guided by these ideas, consider the class $\mathcal{S}$ of Dirichlet series (introduced by Selberg [7]). A Dirichlet series

$$F(s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s}$$

is in $\mathcal{S}$ provided that it satisfies the following hypotheses.

(1) Analyticity: $(s-1)^m F(s)$ is an entire function of finite order for some nonnegative integer m.

(2) Ramanujan Hypothesis: $a_n \ll_{\varepsilon} n^{\varepsilon}$ for any fixed $\varepsilon > 0$.

(3) Functional equation: There must be a function $\gamma_F(s)$ of the form

$$\gamma_F(s) = \varepsilon Q^s \prod_{i=1}^{k} \Gamma(w_i s + \mu_i)$$

where $|\varepsilon| = 1$, $Q > 0$, $w_i > 0$, and $\Re \mu_i \geqslant 0$ such that

$$\Phi(s) = \gamma_F(s) F(s)$$

satisfies

$$\Phi(s) = \overline{\Phi}(1-s)$$

where $\overline{\Phi}(s) = \overline{\Phi(\bar{s})}$.

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