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EVOLVING PLANE CURVES BY CURVATURE IN RELATIVE GEOMETRIES

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0. Introduction. In this paper we study the motion of a plane curve in which the velocity vector field is determined at each point by the curvature of the curve and by the direction of the curve's normal vector. The goal is to describe the asymptotic shape of the curve as it shrinks to a point. The partial differential equation describing the motion can be written

$$X_t = \gamma(\theta)kN \tag{0.1}$$

where X is the position vector of the curve, the subscript t denotes partial differentiation with respect to time, N is the (inward pointing) normal vector to the curve, θ is defined by $N = (-\cos\theta, -\sin\theta)$, and γ is some given function of direction which is smooth and strictly positive.

Our main result is the following.

THEOREM 4.4 (The main theorem). *If γ can be written as*

$$\gamma(\theta) = \frac{\tilde{h}(\theta)}{\tilde{k}(\theta)} \tag{0.2}$$

where $\tilde{h}$ and $\tilde{k}$ are the support function and the curvature respectively of some smooth, symmetric strictly convex body $\tilde{K}$, then every convex curve converges to the shape of $\partial\tilde{K}$ as the curve shrinks to a point. More precisely, the laminae enclosed by the evolving curves, when renormalized to have the same area as $\tilde{K}$ and appropriately translated, will converge to $\tilde{K}$ in the Hausdorff metric.

For $\gamma \equiv 1$, equation (0.1) describes the "curve-shortening" flow which has been studied in a number of papers. The earliest work seems to be due to Mullins [Mu] who proposes the equation as a model for the motion of grain boundaries. Among other things, Mullins found the self-similar solutions to the equation. References to other work on the equation can be found in the metallurgy literature in [VCMS]. The equation was later proposed as a model for finding closed geodesics on surfaces by K. Uhlenbeck and investigated in a series of papers which includes [G1], [G2], [GH], [EG], [Gr1], and [Gr2]. The flow also arises in simplified models of the motion of phase boundaries. This aspect of the problem is reviewed briefly at the end of Section 1 and presented in more detail by Angenent and Gurtin in [AnGu].

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