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ON THE FIRST BETTI NUMBER OF A HYPERBOLIC MANIFOLD WITH AN ARITHMETIC FUNDAMENTAL GROUP

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1. Introduction. The purpose of this paper is to address the following problem. Let Γ be an arithmetic lattice in $SO(n, 1)$. Does there exist a subgroup $\Gamma' \subset \Gamma$ of finite index such that the first Betti number $b_1(\Gamma') \neq 0$? Equivalently, is the abelianization of Γ' infinite? We observe that the problem is solved by Millson [Mila] when the lattice Γ is commensurable with the group of units of a quadratic form over a totally real field which has signature $(n, 1)$ at one real place and is anisotropic at the remaining real places. In particular, it is solved for the noncocompact arithmetic lattices. Indeed, for $n \neq 7$ it is a well-known consequence of [Wei] that every noncocompact arithmetic Γ is commensurable with the group of units of a quadratic form over $\mathbf{Q}$ with signature $(n, 1)$. For $n = 7$ one must also consider the exceptional rational structures on $SO(7, 1)$ due to the existence of extra automorphisms of the Dykin diagram of type D_4. By the classification of semisimple algebraic groups over number fields [Tit], the only exceptional simple groups that could possibly give rise to noncocompact arithmetic lattices in $SO(7, 1)$ are those of types ${}^3D_{4,1}^9$ and ${}^6D_{4,1}^9$ listed on page 58 of [Tit]; the corresponding "Tits diagram" has the simple root in the middle circled. Hence the corresponding group of real points cannot be locally isomorphic to $SO(7, 1)$ since the latter has a Tits diagram with the middle root not circled. Thus for all n a noncocompact arithmetic Γ must be commensurable with the group of units of a quadratic form over $\mathbf{Q}$ with signature $(n, 1)$ and, consequently, contains a congruence subgroup Γ' with $b_1(\Gamma') \neq 0$ by [Mila].

By the above considerations, if Γ is any arithmetic lattice of $O(n, 1)$ other than those considered in [Mila], then it must be cocompact. In fact, if $n \neq 3, 7$, then it must be commensurable with the group of units of an appropriate skew-hermitian form over a quaternion field (see Section 2). This second family of arithmetic lattices exists only when n is odd. They were recently considered in [Li], where it is shown that if $n > 5$ and Γ is such a lattice, then it contains a congruence subgroup Γ' with $b_1(\Gamma') \neq 0$. In this paper we give a new proof of this result covering the new case $n = 5$ as well. The main result is the following one.

THEOREM. *Let Γ be an arithmetic lattice in $SO(n, 1)$, $n \neq 3, 7$. Then Γ contains a congruence subgroup Γ' such that $b_1(\Gamma') \neq 0$.*

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