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MINIMAL CURRENTS, GEODESICS, AND RELAXATION OF VARIATIONAL INTEGRALS ON MAPPINGS OF BOUNDED VARIATION

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1. Introduction. We consider a functional $\mathscr{F}$ of a C^1 mapping u from an open set Ω in $\mathbb{R}^n$ into $\mathbb{R}^m$:

$$\mathscr{F}(u) = \int_\Omega f(x, u(x), \nabla u(x))\, dx, \tag{1.1}$$

with the density function

$$f\colon \Omega \times \mathbb{R}^m \times \mathbb{R}^{nm} \to \mathbb{R}, \qquad f = f(x, y, \xi) \tag{1.2}$$

which is nonnegative, continuous, convex in ξ, and linear growth in ξ. We are interested in extending $\mathscr{F}$ to the space of mappings of bounded variation $BV(\Omega, \mathbb{R}^m)$ in a natural way. It often happens that in variational problems no minimizers exist in the space of C^1 mappings, and so it is necessary to extend $\mathscr{F}$ to a wider class so that minimizers exist. There is a natural but abstract way to extend $\mathscr{F}$ to $BV(\Omega, \mathbb{R}^m)$ which is called the L^1_{loc}-lower semicontinuous *relaxation* of $\mathscr{F}$. It is defined by

$$\overline{\mathscr{F}}(u) = \inf\left\{\liminf_{l\to\infty} \mathscr{F}(u_l);\ u_l \in C^1(\Omega, \mathbb{R}^m),\ u_l \to u \text{ in } L^1_{\mathrm{loc}}(\Omega, \mathbb{R}^m)\right\}$$

for $u \in BV(\Omega, \mathbb{R}^m)$. In other words, $\overline{\mathscr{F}}$ is the greatest lower semicontinuous function on $BV(\Omega, \mathbb{R}^m)$ less than $\mathscr{F}$ on $C^1(\Omega, \mathbb{R}^m)$. The idea of relaxation goes back to H. Lebesgue in the definition of area of nonparametric surfaces. It was strengthened by Serrin [Se1, Se2] and extended to a very general setting by De Giorgi. Since the value of $\overline{\mathscr{F}}$ is implicitly defined, the problem is to find an explicit integral representation of $\overline{\mathscr{F}}$. This problem is posed by De Giorgi [DG] when $m > 1$ and f depends on y. The goal of this paper is to solve this problem for a class of f, which we call isotropic densities (see (1.5)).

Finding a representation of the relaxation is also important in many other context for example in the theory of harmonic maps ([BBC], [GMS3]) and in the theory of elasticity ([GMS2]).

We briefly explain our representation formula. In this section, for simplicity we restrict ourselves to the case when f is positively homogeneous of degree one in ξ

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