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ISOMETRIC EMBEDDINGS OF SURFACES WITH NONNEGATIVE CURVATURE IN $\mathbf{R}^3$

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1. Introduction. In 1916, Hermann Weyl [W] posed the following problem: Consider the two-sphere S^2 and suppose g is a Riemannian metric on S^2 whose Gauss curvature is everywhere positive. Then, does there exist a global C^2 isometric embedding $X: (S^2, g) \to (\mathbf{R}^3, h)$ where h is the standard flat metric on $\mathbf{R}^3$? Necessarily, the image of such an embedding would bound a convex body; so in other words, is there a diffeomorphism from the sphere to a closed convex surface in $\mathbf{R}^3$ with first fundamental form g?

This can be formulated as a problem in partial differential equations, and the existence of a solution of the Weyl problem is equivalent to solving a uniformly elliptic PDE of Monge-Ampére type.

Weyl himself made the first attempt at solving this problem and made substantial progress. His method of attack was the continuity method. A key step involved obtaining a priori estimates for a surface in $\mathbf{R}^3$ with positive Gauss curvature. Weyl only obtained estimates up to second order. However, in order to insure the existence of a C^2 solution, an estimate on the modulus of continuity of the second derivatives was also needed.

In 1953, Nirenberg [N] solved the Weyl problem under the mild smoothness assumption that the metric have continuous fourth derivatives. His result depended on stronger a priori estimates he had derived for a class of uniformly elliptic PDE's. Finally, in 1962, Nirenberg's result was extended to the case of continuous third derivatives of the metric by Heinz [H].

In a completely different approach to this problem, Alexandroff [A] obtained a generalized solution of Weyl's problem as a limit of polyhedra. The regularity of this solution was proved by Pogorelov [P1], [P2].

The question of uniqueness (within a rigid body motion) was also considered by Weyl. That is, are two closed, isometric convex surfaces congruent? This was proved in 1927 by Cohn-Vossen [CV] under the assumption of analyticity of the metric, and later in 1943, this was extended to three times continuously differentiable metrics by Herglotz [He], and finally in 1962 to two times continuously differentiable metrics by Sacksteder [S].

In this paper, the Gauss curvature is assumed to be nonnegative instead of strictly positive, and again one considers the global isometric embedding question. As above, the image of such an embedding, if it exists, bounds a convex body [DL]. However, in this situation, the resulting PDE is degenerate elliptic, and so new

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