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CYCLES IN A PRODUCT OF ELLIPTIC CURVES, AND A GROUP ANALOGOUS TO THE CLASS GROUP.

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1. Introduction. In this paper we consider a subgroup of the second Chow group of a surface, which we call Σ. The group Σ is interesting because it is analogous to the divisor class group of classical number theory. It is defined as follows. Let X be a smooth algebraic surface defined over a number field K and suppose that X has a smooth proper model $\mathscr{X}$ defined over $\mathscr{O}[1/N]$, a localization of the ring of integers in K. Let $j: X \to \mathscr{X}$ be the inclusion of X into $\mathscr{X}$. Then Σ is the kernel of the flat pull-back map j^*:

$$\Sigma := \ker(j^*: \mathrm{CH}^2(\mathscr{X}) \to \mathrm{CH}^2(X)).$$

(As usual, CH^2 denotes the Chow group of codimension-two cycles modulo rational equivalence.) Σ consists of cycle classes supported over finite primes. To see the analogy between Σ and the class group, we need to introduce an exact sequence from algebraic K-theory.

Let $\mathscr{K}_n$ denote the Zariski sheaf associated to $U \mapsto K_n(U)$, where $K_n(U)$ is the nth algebraic K-group of U. Let $\mathscr{X}_p = \mathscr{X} \times \mathbb{F}_p$ be the fibre of $\mathscr{X}$ above p. From the work of Sherman and Bloch there is an exact sequence

$$H^1(\mathscr{X}, \mathscr{K}_2) \to H^1(X, \mathscr{K}_2) \xrightarrow{\partial} \coprod_p H^1(\mathscr{X}_p, \mathscr{K}_1) \to H^2(\mathscr{X}, \mathscr{K}_2) \to H^2(X, \mathscr{K}_2) \to 0$$

where the product is over all primes in $Sp(\mathscr{O}[1/N])$. (By [CTR, §3] or [M, 2.4.1] an element of $H^1(X, \mathscr{K}_2)$ is represented by a formal sum $\sum_D (D, f_D)$, where D is a divisor on X and f_D is a rational function on D satisfying $\sum_D \operatorname{div}(f_D) = 0$. The map ∂ takes $\sum (D, f_D)$ to $\sum \operatorname{div}(f_{\bar{D}})$, where we consider f_D as a function on $\bar{D}$, the closure of D in $\mathscr{X}$.) In our case, since $\mathscr{K}_1 = \mathscr{O}^\times_{\mathscr{X}_p}$ and $H^2(X, \mathscr{K}_2) \simeq \mathrm{CH}^2(X)$, Sherman's exact sequence gives

$$H^1(\mathscr{X}, \mathscr{K}_2) \to H^1(X, \mathscr{K}_2) \to \coprod_{p \in Sp(\mathscr{O}[1/N])} Pic(\mathscr{X}_p) \to \Sigma \to 0. \tag{1.1}$$

If we consider $Sp(K)$ and $Sp(\mathscr{O})$ in place of X and $\mathscr{X}$ and use Sherman's sequence for $\mathscr{K}_1$ (which starts at H^0), we get the well-known sequence

$$\mathscr{O}^\times \to K^\times \to \coprod_p \mathbb{Z} \to Cl(\mathscr{O}) \to 0.$$

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