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LITTLE TOPOLOGY, BIG VOLUME

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The estimation of volume is one of the fundamental techniques of geometry. In particular, estimates for balls are typically an important step in proving finiteness and convergence theorems. The most familiar volume estimates involve curvature hypotheses: If sectional curvature is bounded below (by $\lambda \in \mathbb{R}$), then the volumes of balls of radius r are bounded above (by the volume of the r-ball in the "model" simply connected space form of constant sectional curvature λ). Indeed, the ratio between the volume of the r-ball around a fixed point in the manifold and the "model" r-ball volume is a monotone nonincreasing function with limit 1 as $r \to 0^+$: This is by the standard Rauch comparison theorem for sectional curvature; the result holds for just Ricci curvature bounded below, by a refinement of the Rauch comparison method ([3], [4]). On the other hand, even sectional curvature bounded above does not yield a lower bound on ball volumes, as such; such a lower estimate is valid only inside the injectivity radius, i.e., for r-balls around a point p, with $r \leqslant$ injectivity radius at p.

Being inside the injectivity radius corresponds philosophically to topological simplicity: an open ball of radius $\leqslant$ the injectivity radius is diffeomorphic to the standard Euclidian unit ball. Thus, it becomes natural to formulate a general principle that simplicity of topology is related to lower volume bounds. Rather surprisingly from the viewpoint of classical Riemannian geometry, this general principle turns out to hold in some instances without curvature bounds of any kind. Specifically, Berger [2] proved that there are constants C_n, $n \in \mathbb{Z}^+$, such that, for every compact n-manifold M^n,

$$\operatorname{vol}(M) \geqslant C_n[\operatorname{inj}(M)]^n$$

where $\operatorname{inj}(M)$ = the injectivity radius of M; Berger even obtained a sharp estimate for the constants C_n; he proved that the standard S^n is the unique extremal case for each n (see also [14]). This volume estimate depends on the injectivity radius of the manifold as a whole. Long narrow zeppelins show that the injectivity radius at one point can be large without the volume of the manifold being large.

Corresponding to Berger's "global" volume estimate of the total manifold volume, there is a "local" estimate of the volumes of balls. There are constants C_n,

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