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VERTEX OPERATOR ALGEBRAS ASSOCIATED TO REPRESENTATIONS OF AFFINE AND VIRASORO ALGEBRAS

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Introduction. The first construction of the integrable highest-weight representations of affine Lie algebras or loop algebras by Kac [K] was greatly inspired by the generalization of the Weyl denominator formula for affine roots systems discovered earlier by Macdonald [M]. Though the Macdonald identity found its natural context in representation theory, its mysterious modular invariance was not understood until the work of Witten [W] on the geometric realization of representations of the loop groups corresponding to loop algebras. The work of Witten clearly indicated that the representations of loop groups possess a very rich structure of conformal field theory which appeared in physics literature in the work of Belavin, Polyakov, and Zamolodchikov [BPZ]. Independently (though two years later), Borcherds, in an attempt to find a conceptual understanding of a certain algebra of vertex operators invariant under the Monster [FLM1], introduced in [B] a new algebraic structure. We call vertex operator algebras a slightly modified version of Borcherd's new algebras [FLM2].

Vertex operator algebras are in essence the same as chiral algebras in physics literature (see, e.g., [MS]) though they have never been defined by physicists according to the mathematical standards. Rational conformal field theory can be constructed from the complete sets of irreducible representations of a pair of two identical vertex operator algebras (corresponding to the holomorphic and antiholomorphic sectors) with the equivalent categories of representations. The most important case of conformal field theory built from the representations of two identical vertex operator algebras can be compared to the space of functions on a compact group with the left and right action of the corresponding pair of isomorphic Lie algebras. Vertex operator algebras can in fact be thought of as Lie algebras with an additional complex parameter, which is the source of the geometric realization of conformal field theory by the use of Riemann surfaces [S]. In particular, the three terms of the Jacobi identity for the vertex operator algebras correspond to the three ways of cutting the four-hole sphere into three-hole spheres.

Vertex operator algebras provide a powerful algebraic tool to study the general structure of conformal field theory as well as various specific models. An example of the general properties of vertex operator algebras is the modular invariance of the characters of irreducible representations [Z] which, in the language of conformal field theory, means that the properties of conformal field theory on the sphere

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