Vol. 64, No. 2 DUKE MATHEMATICAL JOURNAL © November 1991

A NOTE ON BOGOMOLOV-GIESEKER'S INEQUALITY IN POSITIVE CHARACTERISTIC

ATSUSHI MORIWAKI

Let k be an algebraically closed field. Throughout this note, we shall fix this field k and every algebraic scheme will be defined over k.

Let X be a d-dimensional nonsingular projective variety with an ample line bundle H. For a torsion-free sheaf Q on X, we set

$$\mu(Q, H) = \frac{(c_1(Q) \cdot H^{d-1})}{\operatorname{rank}(Q)}.$$

Let E be a torsion-free sheaf on X. We say E is *μ-semistable with respect to H* if, for all subsheaves $F \neq 0$ of E, we have

$$\mu(F, H) \leqslant \mu(E, H).$$

For a μ-semistable vector bundle E of rank r, D. Gieseker [4] proved that

$$c_1(E)^2 \leqslant \frac{2r}{r-1} c_2(E)$$

if $\dim X = 2$ and $\operatorname{char}(k) = 0$.

But in the case of positive characteristic his theorem does not hold in general. For example, M. Raynaud [10] constructed a nonsingular projective surface S and an ample line bundle L on S such that $H^1(S, L^{-1}) \neq 0$. For a nonzero element of $H^1(S, L^{-1})$, there is a nontrivial extension

$$0 \to \mathcal{O}_S \to E \to L \to 0$$

of L by $\mathcal{O}_S$. The bundle E has Chern classes $c_1(E) = c_1(L)$ and $c_2(E) = 0$ so that it does not satisfy $c_1(E)^2 \leqslant 4c_2(E)$. On the other hand, by virtue of Mumford's argument (see [11, proof of the Kodaira vanishing theorem in Appendix]) it is easy to see that E is μ-semistable with respect to L.

Nevertheless, N. I. Shepherd-Barron [12] succeeded recently in proving a Bogolomov-type inequality for a vector bundle of rank 2 over a surface considering a purely inseparable covering. In this note we shall prove a Bogomolov-Gieseker–type inequality for a semistable (in the strong sense) vector bundle of any rank. For our proof Maruyama's boundedness of families of torsion-free sheaves [6] is very

Received 6 March 1991.