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INSTANTON INVARIANTS AND FLAT CONNECTIONS ON THE KUMMER SURFACE

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1. Introduction. The new invariants of smooth 4-manifolds, defined by Donaldson [D3, D5] using Yang-Mills moduli spaces, have proved difficult to calculate. With one or two exceptions all calculations have depended on the theorem of [D1], which identifies the moduli spaces of anti–self-dual connections on an algebraic surface with the moduli spaces of stable bundles. The problem of understanding the stable bundles is a difficult one in algebraic geometry, though there has been considerable progress since the publication of [D3].

Amongst the few cases where calculations have been made without recourse to stable bundles, the first is an argument due to Fintushel and Stern [FS3], which establishes that a particular value for a polynomial invariant is nonzero (in fact odd) for the $K3$ surface. This is less information than one obtains from the algebraic geometry, but it is still sufficient for applications. Another case is the recent work of Gompf and Mrowka [GM], where the effect of a surgery (a logarithmic transform) is analysed, using C^∞ techniques. Using this analysis, it was shown in [GM] that by applying logarithmic transforms one can obtain an infinite family of homotopy $K3$ surfaces which are not diffeomorphic to any algebraic surface. This result depends, at bottom, on the fact that a certain integer-valued invariant for the $K3$ surface is nontrivial. The calculation of this last invariant was first made in [D5], using the theorem of [D1], but it has been pointed out to the author by Dieter Kotschick that the use of [D1] can be avoided.

The purpose of this paper is to show how the same integer-valued invariant can be calculated for the $K3$ surface without any serious geometric input beyond our basic understanding of instantons. We shall not use stable bundles and the theorem of [D1], nor shall we use any analysis deeper than the more elementary aspects of Donaldson's theorem on connected sums (as illustrated in Theorem 4.8 of [D5], for example). The argument is not special to $K3$, for we can understand also the homotopy $K3$ surfaces obtained by performing a logarithmic transform. In this way we can reproduce a result first proved by Friedman and Morgan [FM], namely that the fake $K3$ surfaces arising in Kodaira's classification realize infinitely many diffeomorphism types, none of which are diffeomorphic to the real $K3$. We have not extended our calculations to the examples constructed by Gompf and Mrowka, though we indicate one approach in Section 5. In a future paper the author will consider some invariants of elliptic surfaces with larger Euler characteristic, and with enough ingenuity it may be that the argument can be applied elsewhere.

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