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ON ARTHUR'S CLASS EXPANSION OF THE RELATIVE TRACE FORMULA

K. F. LAI

1. Introduction. In a paper of this journal Arthur [1] generalized the Selberg trace formula to an arbitrary reductive group G. His trace formula is an equality of distributions, with the class (or so-called o) expansions on one side and the spectral (or so-called χ) expansions on the other. In Jacquet and Lai we give another generalization of the Selberg trace formula for GL(2); namely, instead of integrating the kernel of the right regular representation $K(x, y)$ over the group G embedded diagonally in $G \times G$, we integrate over another diagonal subgroup H of $G \times G$ instead of G. However, in Jacquet and Lai, as in the classical case of SL(2), we attack the convergence problem of integration directly without the sophisticated means of Arthur. In this paper we propose an analogue of Arthur [1] for the class expansion side of a relative trace formula of an arbitrary reductive group. We hope that this provides an alternative route to study the base change problem proposed by Langlands, in particular to obtain results for compact Shimura varieties where there are no Einsenstein series. For example, this technique allows one to prove Tate's conjecture on algebraic cycles for a family of compact Shimura surfaces (Lai [6]). For other applications and generalisations see Jacquet [5] and Ye [7], [8]. In [3] Hakim refines the results of [4].

As we shall see, once we observe the correct analogue of Arthur's class expansion formula the geometry of the fundamental domain provided by reduction theory in our case is the same as that of Arthur, and so it is simpler than that of the twisted trace formula of the morning seminar of Langlands.

Let F be an algebraic number field and E be a finite Galois extension of F. We write $\mathbb{A}$ for the adèles of F and $\mathbb{A}_E$ for the adèles of E. To simplify notation we use $\mathbf{G}$ to denote an algebraic group defined over F and write G for its group of F rational points, $G_{\mathbb{A}}$ for its group of F adelic points.

Let $\mathbf{G}$ be a connected reductive algebraic group defined over F. Consider $\mathbf{G}$ as an algebraic group over E and apply to it the Weil restriction functor from E to F to obtain $\tilde{\mathbf{G}}$. For each rational character χ of $\tilde{\mathbf{G}}_{\mathbb{A}}$ defined over F, we define the homomorphism $|\chi|$ by

$$|\chi|(x) = \prod_v |\chi(x_v)|_v, \qquad x = \prod_v x_v \in \tilde{\mathbf{G}}_{\mathbb{A}}.$$

The intersection of the kernel of all the $|\chi|$ is denoted by ${}^\circ\tilde{G}_{\mathbb{A}}$. The locally compact group ${}^\circ\tilde{G}_{\mathbb{A}}$ contains $\tilde{G}$ as a discrete subgroup. Let R be the right regular representa-

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