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HIGHER K-THEORY OF THE CATEGORY OF WEAKLY EQUIVARIANT $\mathscr{D}$-MODULES

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In this paper we study the higher K-theory (and the derived category) of weakly equivariant algebraic $\mathscr{D}$-modules on smooth complex algebraic varieties that are weakly equivariant for the action of complex linear algebraic groups. We begin by recalling the basic terminology from [J1], Section 1 and [J3], Section 1. Next, we obtain localisation and Mayer-Vietoris sequences for the higher K-theory of the category of weakly equivariant coherent algebraic $\mathscr{D}$-modules. In the next section we prove an equivariant version of Quillen's theorems on the higher K-theory of graded and filtered rings, equivariant for linear actions by diagonalisable groups. This is applied to conclude that the higher K-theory of the categories of weakly equivariant coherent $\mathscr{D}$-modules and equivariant coherent sheaves are isomorphic when the group is diagonalisable and acts linearly. This completes the proof of the higher Riemann-Roch theorems in [J3]. We conclude by relating the derived category (higher K-theory) of the category of weakly equivariant $\mathscr{D}$-modules with the derived category (higher K-theory, respectively) of the category of weakly equivariant extended $\mathscr{D}$-modules.

1. (1.1) Let X denote a complex smooth quasi-projective algebraic variety provided with the action of a complex linear algebraic group G. In this situation we will assume all the basic terminology from [J3], (1.0)–(1.3). A $\mathscr{D}_X$-module M will mean an $\mathscr{O}_X$-module which is a *left module* (unless we say to the contrary) over the ring $\mathscr{D}_X$. Quasi-coherent (coherent) $\mathscr{D}_X$-modules are defined in the obvious manner. *G-equivariant* (or *weakly G-equivariant*) $\mathscr{D}_X$-modules are defined as in [J3], (1.3). The category of G-equivariant $\mathscr{D}_X$-modules (which are also quasi-coherent, coherent) and G-equivariant maps will be denoted $\mathrm{Mod}^G(\mathscr{D}_X)$ ($\mathrm{Mod}^G_{q\,\mathrm{coh}}(\mathscr{D}_X)$, $\mathrm{Mod}^G_{\mathrm{coh}}(\mathscr{D}_X)$, respectively). Recall ([J1], Section 1 and [J3], Section 1) that these are again abelian categories.

(1.2) We proceed to show that the category $\mathrm{Mod}^G_{q\,\mathrm{coh}}(\mathscr{D}_X)$ is a Grothendieck category; this will follow once we show that each $M \in \mathrm{Mod}^G_{q\,\mathrm{coh}}(\mathscr{D}_X)$ is the filtered colimit of its weakly equivariant coherent $\mathscr{D}_X$-submodules. Let G act on $G \times X$ by translation on the left factor G. Then we observe that $\mathrm{pr}_2^*(\mathscr{D}_X) \cong \mathscr{O}_G \otimes \mathscr{D}_X$ is an equivariant sheaf of rings over $G \times X$ as in [J1], (1.8.0) or [J3] (1.5.0). We may therefore define $\mathrm{Mod}^G_{q\,\mathrm{coh}}(G \times X, \mathrm{pr}_2^*(\mathscr{D}_X))$ in the obvious manner.

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