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INVARIANTS OF CONFORMAL DENSITIES

MICHAEL G. EASTWOOD AND C. ROBIN GRAHAM

1. Introduction. There has long been a rich interaction between questions in geometry and algebraic problems in invariant theory. Riemannian geometry leads to invariant theory for the orthogonal group. This invariant theory, as well as that for the other classical groups, was developed by Weyl [W]. A fundamental element of Weyl's theory and its applications is the complete reducibility of finite-dimensional representations, a consequence of the semisimplicity of the group. Relatively little is known of the corresponding invariant theory for groups which are not semisimple. One such case was considered by Fefferman [F]. Fefferman was interested in understanding the local scalar invariants of CR structures and was led to invariant theory problems for a parabolic subgroup of $\mathrm{SU}(n+1, 1)$. Although he made significant deep progress, many open questions remain. The analogous geometrical construction of invariants for conformal structures was carried out in [FG], but the parabolic invariant theory problems which arise from this have not yet been solved.

In this paper we study the somewhat easier but closely related question of invariants of an auxiliary object on a conformal manifold rather than the invariants of conformal or CR structures themselves. We focus on the simplest case: scalar invariants of conformal densities in conformally flat space. Conformally flat space is the sphere S^n regarded as the homogeneous space G/P, where $G = \mathrm{O}(n+1, 1)/\{\pm I\}$ is the conformal group and the parabolic subgroup P is the stabilizer of a point on S^n. On any smooth n-manifold M, a density of conformal weight $w \in \mathbf{C}$ is a section of the line bundle $|\Lambda^n T^*M|^{-w/n}$. For S^n, an invariant of densities of weight w is a polynomial in the derivatives of the density in a local trivialization with an appropriate invariance property under conformal motions. (See §2 for a precise definition.) Our main results give a construction of such invariants and proofs in certain cases that all invariants arise via our construction. If $w + n/2 \notin \mathbf{N} = \{1, 2, 3, \dots\}$, we give a general construction of invariants of densities of weight w. If in addition $w \notin \mathbf{Z}_+ = \{0, 1, 2, \dots\}$, we show that all invariants arise via our construction. However, if $w \in \mathbf{Z}_+$, one encounters the same fundamental algebraic difficulties which arose in [F]. In fact Fefferman posed and analyzed a family of model algebraic problems, and these very problems arise in our study of invariants of conformal densities. In our notation the problems are to describe the invariants of the P-modules $\mathbf{T}_k$ defined at the beginning of §3. Fefferman's

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