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THE CRITICAL VALUES OF CERTAIN DIRICHLET SERIES ATTACHED TO HILBERT MODULAR FORMS

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Introduction. To define our Dirichlet series, we start with two Hilbert modular forms

$$f(z) = \sum_{\xi \in F} \mu_f(\xi)\mathbf{e}_{\mathbf{a}}(\xi z), \qquad g(z) = \sum_{\xi \in F} \mu_g(\xi)\mathbf{e}_{\mathbf{a}}(\xi z) \tag{1}$$

with respect to a congruence subgroup of $SL_2(F)$, where F is a totally real algebraic number field, $\mathbf{e}_{\mathbf{a}}(\xi z) = \exp(2\pi i \sum_v \xi_v z_v)$ with v running over the set $\mathbf{a}$ of archimedean primes of F, and $z = (z_v)_{v \in \mathbf{a}}$ with variables z_v on the upper half plane H. Let k and l be the weights of f and g, respectively, which are integral or half-integral. Then one of the series to be considered is given in a special case by

$$D(s, f, g) = \sum_{\xi U} \mu_f(\xi)\mu_g(\xi)\xi^{-(k+l)/2} N_{F/\mathbf{Q}}(\xi)^{-s}, \tag{2a}$$

where ξ runs over all the totally positive elements of F modulo a group of units U which is chosen so that each term depends only on ξU. In general, this has no Euler product. We investigate also two types of series with Euler products defined by

$$\mathscr{D}(s, \chi_1, \chi_2) = L'(2s - 2, \Psi_1\Psi_2) \sum_{\mathfrak{a}} \chi_1(\mathfrak{a})\chi_2(\mathfrak{a})N(\mathfrak{a})^{-s}, \tag{2b}$$

$$\mathscr{D}(s, \chi_1^2, \eta) = L'(2s - 2, \eta^2\Psi_1^2) \sum_{\mathfrak{a}} \eta^*(\mathfrak{a})\chi_1(\mathfrak{a}^2)N(\mathfrak{a})^{-s}. \tag{2c}$$

Here χ_1 and χ_2 are systems of eigenvalues in the spaces of forms of weight k and l, respectively, Ψ_μ is the central (Hecke) character of χ_μ, η is a Hecke (idele) character of F, η^* is the ideal character attached to it, $\mathfrak{a}$ runs over all integral ideals, and L' denotes the L-function defined with some finitely many Euler factors removed. Now our main problem concerns the arithmetic nature of the values and the residues of these series (2a, b, c) at some integers or half-integers belonging to a certain interval.

In the present paper we restrict our investigation to the following case:

$$f \textit{ is a cusp form and } k_v \geqslant l_v \textit{ for every } v \in \mathbf{a}. \tag{3}$$

When both k and l are integral, we obtained in [4, 5, 6] some results which may be stated roughly as follows:

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