Vol. 62, No. 3 DUKE MATHEMATICAL JOURNAL © April 1991

# FULLY NONLINEAR OBLIQUE DERIVATIVE PROBLEMS FOR NONLINEAR SECOND-ORDER ELLIPTIC PDE'S

HITOSHI ISHII

**§1. Introduction.** In this paper we are concerned with the fully nonlinear elliptic PDE

$$(1.1) \qquad F(x, u, Du, D^2u) = 0 \qquad \text{in } \Omega.$$

Here $\Omega$ is a bounded open subset of $\mathbb{R}^N$ with $C^1$ boundary, $u$ represents a real unknown function on $\bar{\Omega}$, $F$ is a given real function on $\bar{\Omega} \times \mathbb{R} \times \mathbb{R}^N \times \mathbb{S}^N$, where $\mathbb{S}^N$ denotes the space of real $N \times N$ symmetric matrices with the usual ordering, and $Du$ and $D^2u$ denote the gradient and Hessian matrix of $u$, respectively.

Associated with (1.1) is the fully nonlinear oblique derivative boundary condition

$$(1.2) \qquad B(x, u, Du) = 0 \qquad \text{on } \partial\Omega.$$

Here, by "obliqueness" we mean that $B$ satisfies

$$(1.3) \qquad \langle n(x), D_pB(x, r, p)\rangle > 0 \qquad \text{for } (x, r, p) \in \partial\Omega \times \mathbb{R} \times \mathbb{R}^N,$$

where $\langle \xi, \eta \rangle$ and $D_pB(x, r, p)$ denote the Euclidean inner product of $\xi, \eta \in \mathbb{R}^N$ and the gradient of $B$ with respect to the variable $p$, respectively.

Our basic assumption on $F$ is the degenerate ellipticity. That is, we assume that

$$(1.3) \qquad F(x, r, p, X) \leqslant F(x, r, p, Y) \qquad \text{if } X \geqslant Y$$

for all $(x, r, p) \in \bar{\Omega} \times \mathbb{R} \times \mathbb{R}^N$ and $X, Y \in \mathbb{S}^N$. The strong degeneracy of this ellipticity condition allows (1.1) to cover a fairly large class of PDE's including first-order PDE's. Problem (1.1) and (1.2) thus does not have a classical solution in general, and we here adopt the notion of viscosity solution (see §2) as weak solutions to (1.1) and (1.2).

There is a great deal of literature concerned with oblique derivative problems for (1.1). We refer the reader, if interested in classical approaches to the existence and

Received 20 February 1990. Revision received 3 August 1990.
Author supported in part by Grant-in-Aid for Scientific Research (No. 62540066 and No. 63540072), Ministry of Education, Science and Culture.