Vol. 61, No. 2 DUKE MATHEMATICAL JOURNAL © October 1990

A TAMENESS CRITERION FOR GALOIS REPRESENTATIONS ASSOCIATED TO MODULAR FORMS (MOD p)

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We begin by recalling some results on the 2-dimensional Galois representations which are associated to modular forms (mod p). If $f = \Sigma\, a_n q^n$ is a normalized cuspidal eigenform of weight k and character ε for $\Gamma_1(N)$, with coefficients in a finite field E of characteristic p, there is a continuous semi-simple Galois representation

$$\rho_f\colon \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{GL}_2(E)$$

which is characterized as follows. The representation ρ_f is unramified for all primes $l \nmid Np$, and the matrix $\rho_f(\mathrm{Frob}_l)$ has characteristic polynomial $x^2 - a_l x + \varepsilon(l)l^{k-1}$. The representation ρ_f was conjectured to exist by Serre (cf. [S2], [S6]) and its existence was proved by Deligne (cf. [D1] for the case $N = 1$, and [C] for more general levels). When $k \geqslant 2$ and $a_p \neq 0$, Deligne [D2] also proved that the restriction of ρ_f to a decomposition group at p in $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ has image contained in a Borel subgroup of $\mathrm{GL}_2(E)$. Up to conjugation, this restriction has the form

$$\begin{pmatrix} \chi^{k-1}\cdot\lambda(\varepsilon(p)/a_p) & * \\ 0 & \lambda(a_p) \end{pmatrix} \tag{0.1}$$

where χ is the character of $\mathrm{Gal}(\bar{\mathbb{Q}}_p/\mathbb{Q}_p)$ giving its action on μ_p and, for any $\alpha \in E^*$, $\lambda(\alpha)$ is the unramified character taking Frob_p to α.

In this paper, we will establish a modular criterion conjectured by Serre [S7, pg. 18] for the representation ρ_f to be tamely ramified at p, or more precisely, for $* = 0$ in (0.1). Assume that f has weight $2 \leqslant k \leqslant p$ and $a_p \neq 0$; when $k = p$ assume further that $a_p^2 \neq \varepsilon(p)$, so the two characters $\chi^{k-1}\lambda(\varepsilon(p)/a_p)$ and $\lambda(a_p)$ are distinct. The criterion says that ρ_f is completely reducible when restricted to $\mathrm{Gal}(\bar{\mathbb{Q}}_p/\mathbb{Q}_p)$ if and only if there is a normalized eigenform $g = \Sigma\, b_n q^n$ of weight $k' = p + 1 - k$ and character ε for $\Gamma_1(N)$ over E, whose Fourier coefficients satisfy $n^k b_n = n a_n$ for all $n \geqslant 1$.

The relationship between f and g is symmetric (for example, the relation between Fourier coefficients may be written $n b_n = n^{k'} a_n$), and Serre calls the pair (f, g) of normalized eigenforms "companions". An equivalent formulation of companionship is that the Galois representations ρ_f and ρ_g satisfy: $\rho_f \otimes \chi \simeq \rho_g \otimes \chi^k$. Using this, it is easy to show that the existence of a companion forces ρ_f to be completely

Received October 23, 1989. Revision received February 15, 1990.