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THE KERNEL SPECTRAL SEQUENCE OF VANISHING CYCLES

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Introduction. Let $f: X \to S$ be a projective morphism of a complex manifold X onto an open disc S, and $X_0 = f^{-1}(0)$. Assume f to be smooth over $S^* := S\backslash\{0\}$ (by restricting S if necessary). Let $\psi_f\mathbb{Q}_X$ denote Deligne's vanishing cycle sheaf complex [D1], for which one has an isomorphism $H^j(X_t, \mathbb{Q}) \simeq H^j(X_0, \psi_f\mathbb{Q}_X)$ for any $t \in S^*$ (noncanonical, depending on the choice of a lifting of t to the universal covering of S^*). We have the (second) spectral sequence

$$E_2^{p,q} = H^p(X_0, \mathscr{H}^q\psi_f\mathbb{Q}_X) \Rightarrow H^{p+q}(X_0, \psi_f\mathbb{Q}_X). \tag{0.1}$$

If X_0 is a divisor with normal crossings, $\mathscr{H}^q\psi_f\mathbb{Q}_X$ is easy to calculate, and Illusie had made the following

(0.2) CONJECTURE. *If X_0 has normal crossings,*
(i) *the spectral sequence* (0.1) *degenerates at E_3, and*
(ii) *the induced filtration on $H^j(X_t)$ is the kernel filtration, up to a shift.*

Here, the kernel (resp. image) filtration is the increasing (resp. decreasing) filtration K (resp. I) defined by $K_k = \operatorname{Ker} N^{k+1}$ (resp. $I^i = \operatorname{Im} N^i$) for $k \geqslant -1$ (resp. $i \geqslant 0$) as in [SZ: p. 499], where N is the logarithm of the unipotent part of the monodromy (up to a Tate twist). This conjecture is a generalization of the local invariant cycle theorem [C] (which is the case $q = 0$), and assertion (i) is (for general reasons) equivalent to the E_2-degeneration of the spectral sequence

$$E_1^{-k,j+k} = H^j(X_0, Gr_k^\tau\psi_f\mathbb{Q}_X) \Rightarrow H^j(X_0, \psi_f\mathbb{Q}_X), \tag{0.3}$$

where τ is the canonical filtration ([D2: II, (1.4)]). In the normal crossing case, it can be deduced from [St1–2] that the canonical filtration on $\psi_f\mathbb{Q}_X$ coincides with the kernel filtration of its nilpotent endomorphism ν, and (0.2) (ii) was proved in [Z: (8.25)] by using (indirectly) the monodromical property of the weight filtration (see (0.6) below).

In this note, we prove:

(0.4) THEOREM. *Let $f: X \to S$ be a proper morphism of an irreducible analytic space X onto an open disc S. Assume there exists a proper bimeromorphic morphism*

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