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GEOMETRIC BOUNDS ON THE DENSITY OF RESONANCES FOR SEMICLASSICAL PROBLEMS

JOHANNES SJÖSTRAND

0. Introduction. In this paper, we shall give upper bounds on the number of resonances in certain regions in the complex plane close to the real axis, for semiclassical operators like

$$-h^2\Delta + V(x), \tag{0.1}$$

when h is small.

Many of the phenomena are similar (or can be expected to be similar) to the corresponding ones for the exterior Dirichlet (or Neumann) problem for the Helmoltz equation

$$(\Delta + k^2)u = 0 \quad \text{in } \mathbb{R}^n \backslash A, \tag{0.2}$$

where A is bounded and has a smooth boundary (and n is odd unless otherwise specified). This case has been a source of inspiration for our study of the semiclassical case, and we also believe that the new results presented in this paper for the semiclassical case, will have analogues for (0.2); so, in order to situate our results, it can be useful to recall some known results also for the exterior problem.

For the problem (0.2), the resonances are usually defined in the framework of the Lax-Phillips scattering theory [LPh] as poles of the scattering matrix, but we can also view them as certain complex values, k in the lower half plane, for which (0.2) has a non-trivial solution in a suitable space. When studying the location of resonances, the geometry usually enters in an essential way. If the obstacle is non trapping in the sense that no maximal optical rays in $\mathbb{R}^n \backslash A$, (reflected according to the rules of optics in ∂A,) can be contained in a bounded set, it follows from the results on propagation of $\mathrm{mod}(C^\infty)$ singularities of Melrose-Sjöstrand and Ivrii (see the book of Hörmander [Hö] and references given there) combined with the Lax-Philips theory, that there are only finitely many resonances in a logarithmic neighborhood of the real axis. In the case of nontrapping obstacles with analytic boundary, Bardos-Lebeau-Rauch [BLeR] showed that there can only be finitely many resonances inside a parabolic neighborhood of the real axis of the form $\mathrm{Im}(k) \geqslant -C^{-1}|\mathrm{Re}\, k|^{1/3}$. Under additional assumptions, they also determined the optimal value of the constant C. Again this result is based on a result of propagation of singularities now modulo Gevrey 3, due to Lebeau.

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