EVERY CURVE IS A HURWITZ SPACE

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Introduction. In this paper, we show that any branched covering map $\pi: C \to \mathbb{P}^1$, ramified only over $\{0, 1, \infty\}$, arises from a Hurwitz space. That is, we show that C is the closure of an irreducible component of a Hurwitz scheme H parametrizing a family of branched covers of $\mathbb{P}^1$, branched only over $\{0, 1, \infty, \lambda\}$, where λ is allowed to vary. Moreover, the map $C \to \mathbb{P}^1$ is the restriction of the natural map $H \to \mathbb{P}^1$ corresponding to the position of λ. It then follows by a result of Belyi [B] that every projective curve defined over $\bar{\mathbb{Q}}$ is such a Hurwitz curve. More generally, any $C \to \mathbb{P}^1$ branched over $\{p_1, \ldots, p_b\}$ $(b \geqslant 3)$ must arise from a Hurwitz space parametrizing branched covers of $\mathbb{P}^1$ branched over $\{p_1, \ldots, p_b, \lambda\}$, where the p_i are fixed and λ is allowed to vary.

In section 1 we state the main theorem and some arithmetic applications. Section 2 contains two constructions which we use in the proof of the theorem. The proof itself appears in section 3.

Section 1. Consider the moduli space $H^0_{n,B}$ parametrizing isomorphism classes of (irreducible) n-sheeted branched covers $f: C \to \mathbb{P}^1$ whose branch locus consists exactly of a preassigned finite set $B \subset \mathbb{P}^1$ and of a variable additional point $\lambda \in \mathbb{P}^1 \backslash B$. Here $n > 1$, and two n-sheeted branched covers $f: C \to \mathbb{P}^1$, $f': C' \to \mathbb{P}^1$ are considered isomorphic if there is an isomorphism $g: C \to C'$ satisfying $f = f' \circ g$:

$$\begin{tikzcd} C \arrow[rr, "g"] \arrow[dr, "f"'] & & C' \arrow[dl, "f'"] \\ & \mathbb{P}^1 & \end{tikzcd} .$$

The natural map

$$\pi^0_{n,B}: H^0_{n,B} \to \mathbb{P}^1$$

$$(f: C \to \mathbb{P}^1) \mapsto \lambda \in \operatorname{branch}(f) \backslash B$$

makes $H^0_{n,B}$ into a finite cover of $\mathbb{P}^1 \backslash B$. There is therefore a unique nonsingular complete curves $H_{n,B}$ and a branched covering map $\pi_{n,B}: H_{n,B} \to \mathbb{P}^1$ extending $\pi^0_{n,B}$.

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