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FILTRATIONS OF B-MODULES

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Introduction. Let k be an algebraically closed field of characteristic zero, let G be a simply connected semisimple algebraic group, and let B be a Borel subgroup of G. For every Schubert variety $S \subseteq G/B$, let ∂S be the union of all codimension-1 Schubert subvarieties of S. For every dominant weight λ, set $F_S(\lambda) = \Gamma(S, \mathscr{L}(-\lambda))$, and let $R_S(\lambda)$ be the kernel of $\Gamma(S, \mathscr{L}(-\lambda)) \to \Gamma(\partial S, \mathscr{L}(-\lambda))$.

Let M be a B-module. One says that M is a *strong* module (has an excellent filtration in the terminology of [20]) if M has a filtration whose subquotients are some $F_S(\lambda)$. Similarly, one says that M is a *weak* module (has a relative Schubert filtration in the terminology of [26]) if M has a filtration whose subquotients are some $R_S(\lambda)$. Note that a strong module is automatically a weak module [26].

The goal of this paper is to prove the following two theorems:

THEOREM 1. *Let M be a B-module and let λ be a dominant weight. Then we have*
1. *if M is weak, then $M \otimes k_{-\lambda}$ is weak;*
2. *if M is strong, then $M \otimes k_{-\lambda}$ is strong.*

THEOREM 2. *Let n be an integer, let $M_1, \ldots, M_n$ be n weak modules, and let λ be a weight. Suppose $\lambda - n\rho$ is dominant. Then the B-module $M_1 \otimes \cdots \otimes M_n \otimes k_{-\lambda}$ is strong.*

The remaining part of the introduction is divided into two parts. In the first part, I discuss previous related works. In the second part, I give some details about the proof of the theorems.

A. *On the filtrations of B-modules.* The main motivation for the study of weak and strong filtrations is the study of tensor products of G-modules. Let λ, μ be dominant weights and $L(\lambda)$, $L(\mu)$ the simple G-modules with highest weight λ and μ. Kostant's formula gives the multiplicity of the components of $L(\lambda) \otimes L(\mu)$.

However, it is difficult to use Kostant's formula to prove that some components do appear (actually, it is even difficult to prove combinatorially that the multiplicities are $\geqslant 0$!). Using theorems and the induction functor D from B to G, I can exhibit some concrete subquotients of $L(\lambda) \otimes L(\mu)$. For example, Theorem 1 implies immediately the Parthasaraty–Ranga Rao–Varadarajan conjecture, which was already independently proved by S. Kumar [10] and me [15], [16]. (The $SL(n)$ case is due to P. Polo [20]; later, some special cases were proved again by M. McGovern [17] and P. Littelman (oral communication).) These applications are given in section 7.

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