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THE HAMPWILE THEOREM FOR NONLINEAR EIGENVALUES

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1. Introduction. The mountain pass lemma is a very useful tool in finding stationary points of functionals. A simple situation concerns a continuously differentiable functional $G(u)$ on a Banach space B such that there are $\rho \in \mathbf{R}$, $\delta > 0$, and $e \in B$ such that $\delta < \|e\|$, $G(0) < \rho$, $G(e) < \rho$, $G(u) \geqslant \rho$ for $\|u\| = \delta$. Under these circumstances we wish to conclude that there is some $u \in B \setminus \{0\}$ such that

$$G'(u) = 0. \tag{1.1}$$

This is untrue unless we impose some sort of compactness condition. The most widely known condition is that of Palais-Smale (PS), which requires that $G(u_k) \to c$, $G'(u_k) \to 0$, imply that $\{u_k\}$ have a convergent subsequence. This condition has the drawback that it requires sets of the form $|G(u) - c| < \varepsilon$, $\|G'(u)\| < \varepsilon$, to be bounded for $\varepsilon > 0$ sufficiently small. Substitutes for the PS condition which do not make this stipulation have been proposed by several authors [3, 5], but they must deal with $G(u)$ and $G'(u)$ in unbounded regions in B. Recently, the author [16] has introduced a method which requires only a knowledge of $G(u)$ and $G'(u)$ on a bounded region. In order to accomplish this, the author used boundary conditions on parts of the boundary.

A similar question can be raised if we are searching not for a solution of (1.1) but for an eigenelement, i.e., a $u \in B \setminus \{0\}$ and a $\beta \in \mathbf{R}$ such that

$$G'(u) = \beta u. \tag{1.2}$$

Since we are asking for less, we would expect to be able to relax some of the requirements made in the mountain pass lemma. The purpose of the present paper is to show that indeed this is the case. In fact, we shall show that we need only consider $G(u)$ and $G'(u)$ in a ball of radius $\|e\|$ and use a local compactness criterion. No boundary conditions are required in this case even though they play a role in the proof. Moreover, we remove the requirement that $G(0) < \rho$. All of these improvements strengthen the applicability of the theorems. We present some applications here.

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