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DISTINGUISHED CUSP FORMS ARE THETA SERIES

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Let k be a number field. Consider the dual pair (Sp_{2n}, O_m) over k. By a well-known procedure, one may lift cusp forms on O_m to forms on Sp_{2n} (the theta lifting). For $m < n$, the lifted forms are singular and therefore cannot be cuspidal. (See Lemma 1.1 below.) Things start to be more interesting when $m = n$, where sometimes the lifted forms can be cuspidal. Included among these are the famous examples constructed by Howe and Piatetski-Shapiro [8]. Such cusp forms provide, in some sense, the worst kind of violations of the Ramanujan-Petersson conjecture.

Assume then $m = n$. The theta liftings alluded to above enjoy a distinguished property: They have relatively few nondegenerate Fourier coefficients along the so-called Siegel parabolic subgroup of Sp_{2n} (see Definition 1.3 for a more precise statement). Our purpose in this paper is to show that this rather elementary property actually characterizes these theta-lifted cusp forms. The reader may view our work here as generalizations of the paper of Gelbart and Piatetski-Shapiro [2], where the authors characterized distinguished forms on the metaplectic cover of Sp_2 $(= SL_2)$ as theta series.

The proof of our main result (Theorem 1.5) is not difficult. It amounts essentially to putting together two ingredients: one local (Proposition 2.2), which can be viewed as the local version of Theorem 1.5; one global, which is the L-function characterization of theta series due to Piatetski-Shapiro and Rallis [12].

Thus we now have two characterizations of theta series: one in terms of L-functions and the other by means of Theorem 1.5. As pointed out by the referee, for $n = 2$ there is yet another characterization, by Soudry [16]: theta series as CAP representations.

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§1. The notion of a distinguished cusp form. Let $\mathbf{A}$ be the ring of adeles of k. Fix a nontrivial character ψ of $\mathbf{A}$, trivial on k. Consider the following unipotent subgroup N of Sp_{2n}:

$$N = \left\{ \begin{pmatrix} I_n & B \\ 0 & I_n \end{pmatrix} \middle| B \text{ is an } n \times n \text{ symmetric matrix} \right\} \tag{1}$$

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