Vol. 58, No. 3 DUKE MATHEMATICAL JOURNAL © June 1989

CLASSIFICATION OF SINGULAR SOLUTIONS OF A NONLINEAR HEAT EQUATION

S. KAMIN, L. A. PELETIER, AND J. L. VAZQUEZ

0. Introduction. In this paper we consider the problem of finding all nonnegative solutions of the nonlinear heat equation with absorption

$$u_t = \Delta(u^m) - u^p \quad \text{in} \quad Q = \mathbb{R}^n \times (0, T) \tag{0.1}$$

for $n \geqslant 1$, $m, p > 1$, with initial conditions

$$u(x, 0) = 0 \quad \text{for} \quad x \neq 0. \tag{0.2}$$

Here we mean by a solution a function $u(x, t)$ which is defined, nonnegative and continuous in $\bar{Q}\backslash\{(0, 0)\}$, satisfies (0.1) in the sense of distributions, (0.2) in the classical sense, and is uniformly bounded in x for every $t \in (0, T)$. The behavior of $u(x, t)$ as $(x, t) \to (0, 0)$, $(x, t) \in Q$ is not prescribed so that u may exhibit a singularity at the origin. A first example of a solution of (0.1), (0.2) is the trivial solution $u \equiv 0$. Nontrivial solutions can be obtained by considering the problem

$$(P_c)\begin{cases} u_t = \Delta(u^m) - u^p & \text{in} \quad Q, \\ u(x, 0) = c\delta(x) & \text{in} \quad \mathbb{R}^n, \end{cases}$$

where δ is Dirac's mass and $c \in \mathbb{R}^+$. A solution u_c of P_c is called a *fundamental solution* (*of value* c) for equation (0.1). Fundamental solutions satisfy the property

$$\lim_{t \to 0} \int_{|x|<r} u(x, t)\, dx = c \tag{0.3}$$

for every $r > 0$. Brezis, Peletier and Terman [BPT] found another type of solution of (0.1), (0.2) in the case $m = 1$, $1 < p < (n + 2)/n$ which has a stronger singularity at (0, 0), i.e., such that

$$\lim_{t \to 0} \int_{|x|<r} u(x, t)\, dx = +\infty. \tag{0.4}$$

These solutions are called *very singular solutions.*

Received May 6, 1988. Revision received November 7, 1988.