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A SHORT NOTE ON THE EVOLUTION OF A SURFACE BY ITS MEAN CURVATURE

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1. Introduction. Anyone who has blown soap bubbles knows that if you don't blow hard enough, the soap film deforms back to its equilibrium position and no bubble forms. One model for this deformation involves moving the surface by its mean curvature vector. Of course, as a physical model, this curvature flow is a great simplification, but it has many fascinating geometrical properties. In particular, it reduces the area of the surface as fast as possible, in the sense that it is the gradient flow for the area functional. The question immediately arises, is it possible to blow bubbles in this model? In other words, can an embedded surface flowing by its mean curvature vector change its topology without shrinking to a point?

In [8], Huisken shows that a convex hypersurface in $\mathbf{R}^n$, $n \geqslant 3$, shrinks smoothly to a point, getting round in the limit. This generalizes the result of Gage and Hamilton [5] for convex plane curves. Nonembedded curves in the plane can develop singularities before they shrink to points [1], [2], but in [6] the author shows that embedded plane curves always shrink smoothly until they are convex, and then they shrink smoothly to points by the evolution theorem for convex curves. Thus two-dimensional creatures cannot blow soap bubbles!

We will present geometric criteria which guarantee that a surface embedded in $\mathbf{R}^3$ develops a singularity before it can shrink to a point. In particular, we confirm the suspicion, strongly supported by the computer studies of Sethian [10] and others, that a sufficiently long and narrow barbell handle will pinch off quickly, leaving two contracting bubbles.

2. The Barriers. If r is a positive number and if $l > \pi r/2$, then consider the function $f(x)$ defined by

$$f(x) = \begin{cases} r\cosh\left(\dfrac{|x| - l}{r}\right), & \text{if } |x| > l; \\ r, & \text{otherwise.} \end{cases}$$

Rotating the graph of $f(x)$ about the x-axis yields a surface of everywhere non-negative mean curvature. Let D be the solid of revolution consisting of all points between the x-axis and this surface. Finally, let R be any constant satisfying

$$R^2 > \frac{8lr^2}{2l - \pi r}.$$

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