Vol. 58, No. 2 DUKE MATHEMATICAL JOURNAL © April 1989

SINGULARITIES AND RANK ONE PROPERTIES OF HESSIAN MEASURES

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§1. Introduction. We consider a symmetric matrix of Radon measures $(u_{ij})_{1 \leqslant i,j \leqslant N}$ which is the Hessian of a function $u(x)$ defined on an open set Ω in $\mathbb{R}^N$. We are interested in structure of (u_{ij}) on S where (u_{ij}) is not absolutely continuous with respect to the Lebesgue measure.

Our main goal of this paper is to study rank properties of u_{ij} on S. We shall prove that on a nontrivial subset G of S the matrix (u_{ij}) has rank one as a matrix of measures (Theorems A and B). For example we show that if $K \subset S$ has finite q-dimensional Hausdorff measure for some $q < N$, i.e., $H_q(K) < \infty$, then (u_{ij}) has rank one on K up to a H_q measure zero set (shortly, H_q-a.e.). We conjecture that even on S the matrix (u_{ij}) has rank one, however this remains unproved. A simple but typical example is the function

$$u(x) = a_1|x_1| + a_2x_2 + \cdots + a_Nx_N, \qquad a_j \in \mathbb{R}, \qquad 1 \leqslant j \leqslant N$$

with $a_1 \neq 0$. We easily observe that S equals $\{x_1 = 0\}$ and that $u_{ij} = D_iD_ju = 0$ unless $i = j = 1$ and $u_{11} = a_1\delta(x_1)$, where $D_i = \partial/\partial x_i$ and δ is Dirac's delta measure. This implies that (u_{ij}) has rank one on S and has rank zero outside S. Although S has here the Hausdorff dimension $N - 1$, in general S may have a fractional Hausdorff dimenion larger than $N - 1$ (the same occurs with G). In this example $G = S$ and as a matter of fact G always contains (Theorem A)

$$S_q = \{x \in \Omega; 0 < \Theta^{*q}([D^2u], x) < \infty\} \tag{1}$$

up to $[D^2u]$-measure zero set provided $q < N$. Here $[D^2u]$ is a modulus measure of $D^2u = (u_{ij})$ and $\Theta^{*q}(v, x)$ is the upper density function defined by

$$\Theta^{*q}(v, x) = \limsup_{r \downarrow 0} v(B_r(x))r^{-q},$$

where $B_r(x)$ is a closed ball of radius r with center x.

The class of functions u whose Hessian is a matrix of Radon measures is important both in mathematics and physics. As is known by Dudley [10], a Schwartz' distribution u on $\mathbb{R}^N$ is a convex function if and only if (u_{ij}) is a positive semi-definite matrix of measures. In particular, all convex functions belong to this class. Also, as is pointed out in [2], if we consider the smectic state of liquid crystal problem, the

Received August 12, 1987. Revision received June 7, 1988.