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TRANSLATES OF FUNCTIONS OF TWO VARIABLES

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0. Introduction. For $f \in L^1(\mathbf{R}^2)$ and $x \in \mathbf{R}^2$, introduce the translation operator

$$T_x f(t) = f(t - x), \qquad t \in \mathbf{R}^2.$$

Let $\mathbf{R}_+ = (0, \infty)$ and $\mathbf{R}_+^2 = \mathbf{R}_+ \times \mathbf{R}_+$, and regard $L^1(\mathbf{R}_+^2)$ as a closed subspace of $L^1(\mathbf{R}^2)$ by extending the functions to vanish on $\mathbf{R}^2 \setminus \mathbf{R}_+^2$. If f is a function in $L^1(\mathbf{R}_+^2)$, let $I(f)$ be the closure of the linear span of the combined right and upper translates $T_x f$, $x \in \mathbf{R}_+^2$, of f. The aim of the present paper is to attempt to solve the following problem, which was raised by B. Ya. Levin in the late 1950s, according to Boris Korenblum, and recently appeared in [Lev]:

LEVIN'S PROBLEM. *Describe the cyclic vectors of $L^1(\mathbf{R}_+^2)$, that is, characterize those functions $f \in L^1(\mathbf{R}_+^2)$ for which $I(f) = L^1(\mathbf{R}_+^2)$.*

The one-dimensional analog of this problem was solved by Bertil Nyman in his 1950 thesis [Nym] and later independently by V. P. Gurariĭ and B. Ya. Levin [GuL]: the right translates $T_x f$, $x \in \mathbf{R}_+$, of a function $f \in L^1(\mathbf{R}_+)$ span a dense subspace of $L^1(\mathbf{R}_+)$ if and only if

(a) $\hat{f}(z) = \int_0^\infty e^{-tz} f(t)\, dt \neq 0$ for all $z \in \overline{\Pi}_+ = \{w \in \mathbf{C}: \operatorname{Re} w \geqslant 0\}$, and
(b) f does not vanish almost anywhere on any interval $(0, \alpha)$, $\alpha > 0$.

Since both of the above references are somewhat inaccessible, we refer the interested reader to Garth Dales's survey article [Da1], where a proof is given (pp. 196–201), and Gurariĭ's monograph [Gur]. Judging from Nyman's result, one might guess that the right condition in our two-dimensional situation is that

(a′) $\hat{f}(z_1, z_2) = \int_0^\infty \int_0^\infty e^{-t_1 z_1 - t_2 z_2} f(t_1, t_2)\, dt_1\, dt_2 \neq 0$ for all $(z_1, z_2) \in \overline{\Pi}_+^2 = \overline{\Pi}_+ \times \overline{\Pi}_+$, and
(b′) f does not vanish almost everywhere in any neighborhood of the origin.

Clearly (a′) and (b′) are necessary. However, they are far from sufficient. Namely, there are other types of conditions that remain invariant under right and upper translations, too; for instance,

$$\int_0^\infty f(t_1, t_2)\, dt_1 = 0 \quad \text{for almost all} \quad t_2 \in (0, 1),$$

is one.

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